Contents

Chapter 1

Basic Concepts

1.1 Dynamics: History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytic study of planetary motion
1890	Poincare	Geometri approach , nightmare of chaos
1920-1950		Nonlinear oscillators in physics and engineering, invation of radio, rador, laser
1920-1960	Birkhoff Kolmogorov Arnol'd Moser	Complex Behavior in Hamiltonial mechanics
1963	Lorenz	Strange Attaractor in simple model of convection
1970s	Ruselle and Takens	Turbalence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universally and renormalization connection between and phase transition
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators and their applications

1.2 Dynamical Systems: Definition

A dynamical system is a way of describing the passage in time of all points of a given space S, S is the space of states of some physical system. Mathematically, S might be a Euclidean space or an open subset of Euclidean space or some other space such as a surface in $\mathbb{R}^3$. When we consider dynamical systems that arise in mechanics, the space S will be the set of possible positions and velocities of the system.

Given an initial position $X \in \mathbb{R}^n$, a dynamical system on $\mathbb{R}^n$ tells us where X is located 1 unit of time later, 2 units of time later, and so on. We denote these new positions of X by X_1, X_2, and so forth. At time zero, X is located at position X_0. One unit before time zero, X was at X_{-1}.

In general, the trajectory of X is given by X_t . If we measure the positions X_t using only integer time values, we have an example of a discrete dynamical system, If time is measured continuously with $t \in \mathbb{R}$, we have a continuous dynamical system. If the system depends on time in a continuously differentiable manner, we have a smooth dynamical system.
Physically, a dynamical system is an object or collection of objects in the real world which evolves in time.

1. A fluid in a container subjected to stirring or external influences such as changes in temperature or pressure
2. The population at time t of a certain species of animal or plant
3. The current through a wire (motion of electrons)
4. The motion of a object suspended by a spring or rigid rod pendulum
5. Molecules of a gas in a container

1.3 Elementary Examples

1.3.1 Discrete Dynamical System

Consider a bank account opened with 100 at 6 per interest compounded annually. The state of this system at any instant in time can be described by a single number: the balance in the account. In such case time is discrete. That is to say, time is a sequence of separate chunks each following the next like beads on a string. For the bank account, it is easy to write down the rule which takes us from the state of the system at one instant to the state of the system in the next instant, namely,

$$x(k+1) = 1.06x(k) \tag{1.1}$$

where $x(k)$ denotes the state of the system at time k. In this example (since interest is only paid once a year) time is always a whole number. A complete description of the system is

$$x(k+1) = 1.06x(k),$$
$$x(0) = 100.$$

It is customary to begin time at 0, and to denote the initial state of the system by x_0. In this example $x_0 = x(0) = 100$.
The state of the bank account in all future years can now be computed. We see that $x(1) = 1.06x(0) = 1.06100 = 106$, and then $x(2) = 1.06x(1) = 1.06106 = 112.36$. Indeed, we see that

$$x(k) = (1.06)^k 100,$$

or more generally,

$$x(k) = 1.06^k x_0. \tag{1.2}$$

Therefore, $1.06^k x_0$ is a general formula for $x(k)$. However, we can verify that equation (1.2) is correct by checking two things:

1. that it satisfies the initial condition $x(0) = x_0$, and
2. that it satisfies equation (1.1).

That is

$$x(0) = (1.06)^0 x_0 = x_0$$

and

$$x(k+1) = 1.06^{k+1} x_0 = (1.06)(1.06)^k x_0 = 1.06x(k).$$

A larger context
Let us put this example into a broader context which is applicable to all discrete time dynamical systems. We have a state vector $x \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$

$$x(k+1) = f(x(k)).$$

In the example,$n = 1$ (the bank account is described by a single number: the balance) and the function $f : \mathbb{R} \to \mathbb{R}$ is simply $f(x) = 1.06x$. Later, we consider more complicated functions f. Once we are given that $x(0) = x_0$ and $x(k+1) = f(x(k))$, we can, in principle, compute all values of $x(k)$, as follows:

$$\begin{aligned}
x(1) &= f(x(0)) = f(x_0) \\
x(2) &= f(x(1)) = f(f(x_0)) \\
x(3) &= f(x(2)) = f(f(f(x_0))) \\
x(4) &= f(x(3)) = f(f(f(f(x_0)))) \\
&\vdots \\
x(k) &= f(x(k-1)) = f(f(\cdots(f(x_0))\cdots))
\end{aligned}$$

where in the last line we have f applied k times to x_0 . We need a notation for repeated application of a function. Let us write $f^2(x)$ to mean $f(f(x))$, $f^3(x) = f(f(f(x)))$, and in general, write

$$f^k(x) = f(f(f(\cdots f(x)\cdots))$$

1.3.2 Continuous Dynamical System

Here, we try to study the continuous dynamical system.
A ball thrown upwards with velocity v reaches a height h. If we know these two numbers, h and v, the fate of the ball is completely determined. The pair of numbers (h, v) is a vector **x** representaing instantaneous status of the ball.
Instead of k in discrete case we use t to denote time, where $t \geq 0$, we start at time $t = 0$. Since we cannot write down a rule for the next instant of time, we instead describe how the system is changing at any given instant. First, if our ball has (upward) velocity v, then we know that $dh/dt = v$; this is the definition of velocity. Second, gravity pulls down on the ball and we have $dv/dt = -g$ where g is a positive constant. So the change in the system is described by

$$h'(t) = v(t) \tag{1.3}$$
$$v'(t) = -g, \tag{1.4}$$

which can be expressed as

$$\begin{bmatrix} h'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{1.5}$$

Since

$$\textbf{x(t)} = \begin{bmatrix} h(t) \\ v(t) \end{bmatrix} \tag{1.6}$$

and

$$x' = f(x) \tag{1.7}$$

we have from (1.5)

$$\textbf{f(x)} = \mathbf{Ax} + \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{and } \mathbf{b} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

Indeed, equation (1.7) is the form for all continuous time dynamical systems. A continuous time dynamical systems has a state vector **x**(t) $\in \mathbb{R}^n$ and we are given a function

$f : \mathbb{R}^n \to \mathbb{R}^n$ which specifies how quickly each component of **x**(t) is changing, i.e., $x_0(t) = f(x(t))$, or more succinctly, $x_0 = f(x)$. Solving the equation we get

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2$$

$$v(t) = v_0 - gt$$

describe the motion of the ball with initial conditions $x_0 = (h_0, v_0)$

1.4 Linear and Non Linear Dynamical System

We can think of a dynamical system as the time evolution of some physical system, such as the motion of a few planets under the influence of their respective gravitational forces. Usually we want to know the fate of the system for long times, for instance, will the planets eventually collide or will the system persist for all times?
Suppose $x = (x_1 \cdots x_n)$ is a point in n-dimensional space R^n that traces out a curve through time. We can describe this as

$$x = x(t) = (x_1(t) \cdots x_n(t) \text{ for } -\infty < t < \infty'$$

The rate and direction of change of $x(t)$ in some region of R^n is

$$\dot{x} = f(x) \; x \in \mathbb{R}^n \tag{1.8}$$

where the dot indicates the derivative with respect to t , so $\dot{x} = dx/dt$. We always assume f has continuous partial derivatives. If we write these vector equations out in full, we get

$$\begin{aligned} \dot{x_1} &= f_1(x_1 \cdots x_n). \\ \dot{x_2} &= f_2(x_1 \cdots x_n). \\ &\vdots \\ \dot{x_n} &= f_n(x_1 \cdots x_n). \end{aligned} \tag{1.9}$$

We call this a set of first-order ordinary differential equations in n unknowns. It is first-order because no derivative higher than the first appears. It is ordinary as opposed to partial because we want to solve for a function of the single variable t , as opposed to solving for a function of several variables. We call $\mathbf{x}(t)$ a **dynamical system** if it satisfies such a set of ordinary differential equations, in the sense that $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ for t in some (possibly infinite) interval.
Here, the variables $x_1, x_2, \cdots x_n$ may represent the concentration of the chemicals in a reactor, the population of the species in an ecosystem or the position and velocity of the planets in the solar system. For example the damped oscillator

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{1.10}$$

can be expressed in the form

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= -\frac{b}{m}x_2 - \frac{k}{m}x_1\end{aligned}$$

is a linear system.
The swinging of a pendulum is governed by the nonlinear equation

$$\ddot{x} + \frac{g}{L}x = 0$$

where x is the angle of the pendulum from the vertical, g the acceleration due to gravity, and L is the length of the pendulum. The equation is expressed as

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= -\frac{g}{L}\sin x_1.\end{aligned}$$

is a nonlinear system. Nonlinearity makes the system very difficult to solve analytically. Considering small angle approximation, that is $\sin x \approx x$ for $(x << 1)$, we get the linear form, which can be solved exactly.

1.5 Flows on the Line

(1.9) is a n-dimensional system. $\dot{x} = f(x)$ is a **one dimensional** or **first order systems**. Here the systems means the dynamical systems, not the set of equations. Also, we consider f is independent of time.
Pictures are more helpful than the formulas for analysing the nonlinear systems.

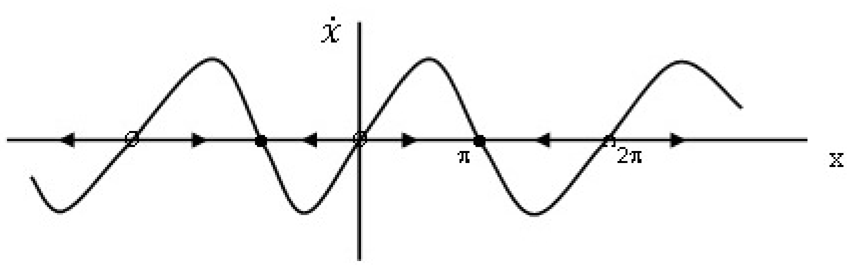
Example 1.5.1.

$$\dot{x} = \sin x \tag{1.11}$$

has solution

$$t = \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \tag{1.12}$$

for $x = x_0$ at $t = 0$. To study the behavior of $x(t)$ it seems not easy as $t \to \infty$



We think of t

as time, x the position of the imaginary particle moving along the real line, and $\dot{x}$ as the velocity of the particle. Then the differential equation $\dot{x} = \sin x$ represents a **vector field** on the line: it dictates the velocity vector $\dot{x}$ at each x. In the figure arrow points to the right for $\dot{x} > 0$ and left for $\dot{x} < 0$.
We imagine that the fluid is flowing steadily along the x-axis with a velocity that varies from place to place according to the rule $\dot{x} = \sin x$.
At the points where $\dot{x} = 0$, there is no flow, such points are called **fixed points**. In the figure, the solid black dots represent **stable** fixed points(also known as **attaractors** or **sinks**) and the open circle circles represent **unstable** fixed points (also known as **repellers** or **sources**). In this way the situation to the given differential equation can be described.

1.6 Fixed Points and Stability

From the example before (see figure) we define the followings,:
The imaginary fluid flowing along the real line with a local velocity $f(x)$ is called the **phase fluid** and the real line is the **phase space**. The imaginary particle is known as **phase point**. The picture ia known as **phase potrait**. The appearence of the phase portrait is controled by the fixed points, x^* defined by the $f(x^*) = 0$, from which we get fixed points or stagnation points of the flow. The solid black dot represents **stable fixed point** (the local flow is toward it) and the open dot is an **unstable fixed point** (the flow is away from it).

1.7 Linear Stability Analysis

The qualitative measure of stability, that is, the rate of decay to a stable fixed point, can be obtained by linearizing about a fixed point.
If x^* is a fixed point, and let $\eta = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. That is,

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x}$$

as x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Using Taylor's expansion we obtain

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2).$$

where $O(\eta^2)$ denotes quadratically small terms in η. Therefore, as $f(x^*) = 0$ as x^* is a fixed point,

$$f(x^* + \eta) = \eta f'(x^*) + O(\eta^2)$$

Now if $f(x^*) \neq 0$, $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2)$$

This is a linear equation in η and is called the **linearization about** x^*. It shows that the perturbation $\eta(t)$ grows exponentially if $f'(x^*) > 0$ and decays if $f'(x^*) < 0$. If $f'(x^*) = 0$, the $O(\eta^2)$ are not negligible and a nonlinear analysis is needed to determine stability.

Example 1.7.1. *Consider the linear equation:*

$$\dot{x} = ax, x(0) = k, \tag{1.13}$$

whose solution depends on a.

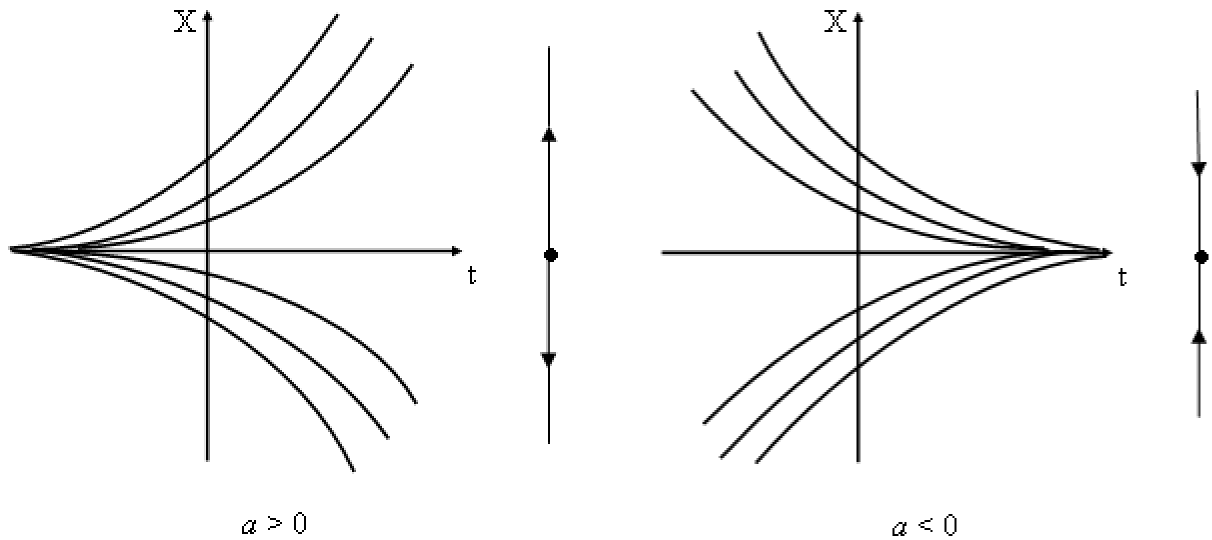
We have

1. If $a > 0, ke^{at}$ becomes ∞, when $k > 0$, and $-\infty$ when $k < 0$ as $t \to \infty$.

2. If $a = 0, ke^{at} = k$, the constant.

3. If $a < 0, ke^{at} = 0$ as $t \to \infty$.

The qualitative behavior is shown in the adjoining figure. The behavior of the solution is quite different for $a > 0$ and $a < 0$. When $a > 0$, all positive solutions tend away from the equilibrium point at 0 as t increases, whereas when $a < 0$, solution tend toward the equilibrium point. We say that the equilibrium point is a **source** when nearby solution tend away from it. The equilibrium point is a **sink** when nearby solutions tend toward it.
We also describe solutions by drawing them on the **phase line**. As $x(t)$ is a function of time, we may view $x(t)$ as a particle moving along the real line. At the equilibrium point, the particle remains at rest (indicated by a solid dot), while any other solution moves up the x-axis as indicated by the arrows in figures.
The equation $\dot{x} = ax$ is **stable** if $a \neq 0$. That is, if a is replaced by another constant b whose sign is the same as a, then the qualitative behavior of the solutions does not change. But if $a = 0$, the slighest change in a leads to a radical change in the behavior of the solutions. We therefore say that we have a **bifurcation** at $a = 0$ in the one parameter family of equation $\dot{x} = ax$.



The equation $\dot{x} = ax, a > 0$ represent is a simplest model for the population growth, where $x(t)$ measures the population of some species at time t. The equation tells that the growth rate of population is directly proportional to the size of the population. Such model does not consider the different circumstances like, famine, diseases, war etc, which are the bounds in the increment of the population. To describe the circumstances, Logistic model is used.

Example 1.7.2. *The Logistic Population Model*

The model considers the followings:

1. If the population is small, the growth rate is nearly directly proportinal to the size of the population.
2. If the population is too large, the growth rate becames negative.

The differential equation satisfying the assumptions is

$$\dot{x} = ax(1 - x/N), \tag{1.14}$$

where $a > 0$ is a parameter, gives the rate of population growth when x is small, while $N > 0$ is a parameter, represents carrying capicity of the population.
If x is small,

$$\dot{x} = ax$$

If $x > N, \dot{x} < 0$.
Without loss of generality, let $N = 1$, then

$$\dot{x} = ax(1 - x) \tag{1.15}$$

is a first order, autonomous, nonlinear equation. The solution of this equation is

$$x(t) = \frac{Ke^{at}}{1 + ke^{at}}$$

where K is determined at time $t = 0$ as

$$K = \frac{x(0)}{1 - x(0)}$$

$$\therefore \; x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}}$$

The qualitative behavior is described by the adjoining figure. In the figure, all solutions,

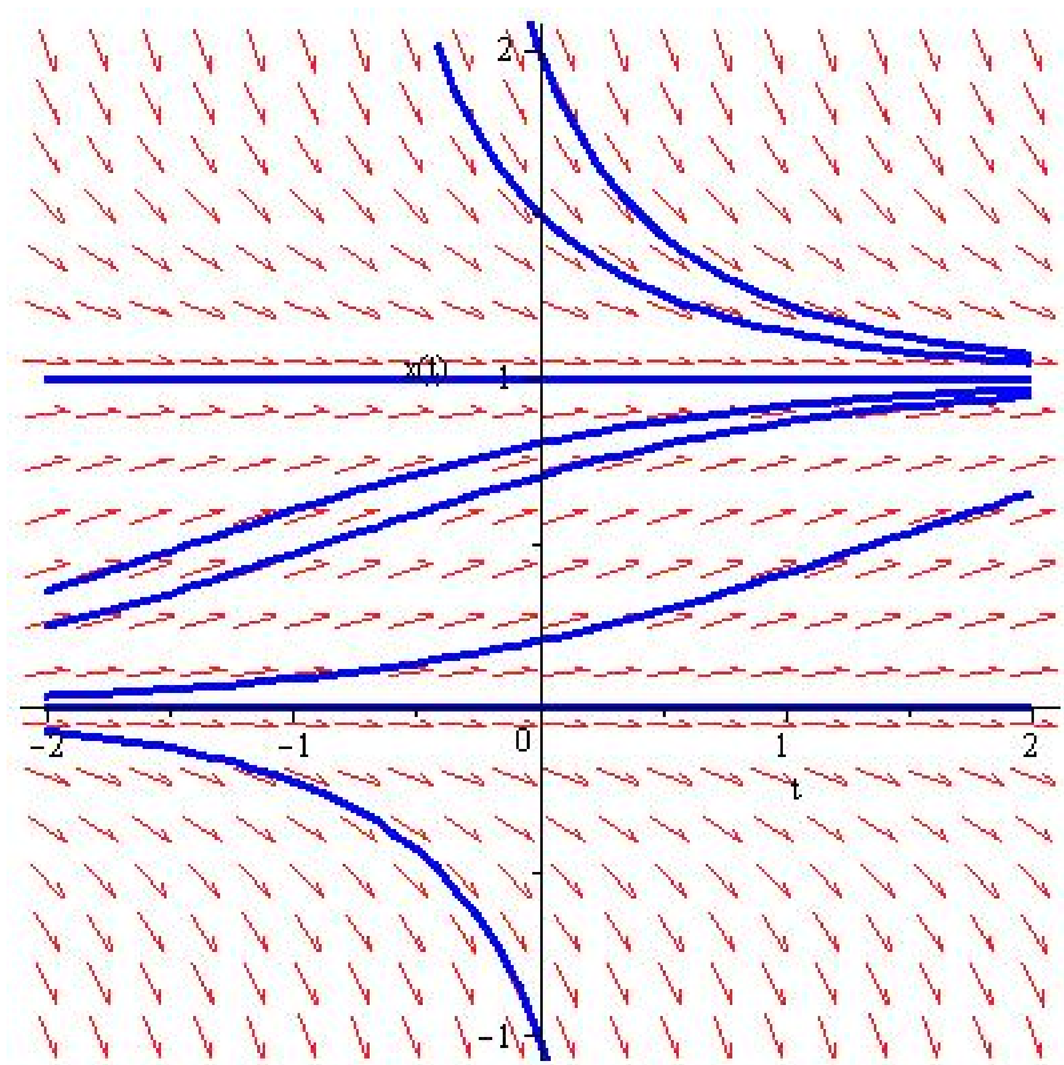
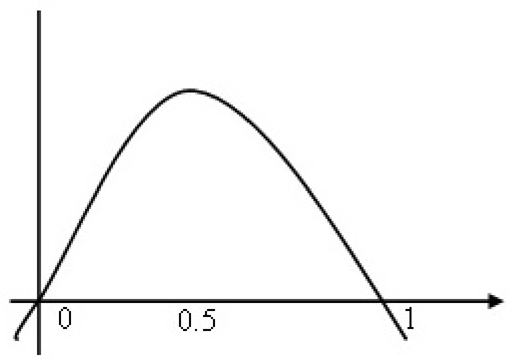


Figure 1.1: The slope field and solution graphs

for which $x > 0$ tend to the population $x(t)v \equiv 1$ and for $x < 0$, the solutions tend to $-\infty$, which is not relevent in the population model.
Also, the qualitative behavior can be described by the figure:



graph crosses the x - axis at the two points $x = 0$ and $x = 1$, which represent the equilibrium points. When $0 < x < 1$, we have $f(x) > 0$. Hence slopes are positive any point (t, x) with $0 < x < 1$, so solutions must increase in this region. We have for $x > 1$ and $x < 0$, $f(x) < 0$, so solution must decrease in these regions.
Here, $x = 0$ is a source and $x = 1$ is a sink. Near 0, we have $f(x) > 0$, if $x > 0$ so slopes are positive and solution increase. But, if $x < 0$, then $f(x) < 0$ and hence the solutions decrease. Thus, nearby solutions always move away from 0 and so 0 is a source. Similarly 1 is a sink.

Exercises 1.

1. Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

2. Determine the stability of the fixed point for $\dot{x} = \sin x$.

3. Classify the stability and draw the graph of $\dot{x} = 4x^2 - 16$.

4. Using the linear stability analysis, determine the stability of the fixed points for $\dot{x} = \sin x$.

5. Classify the fixed points of the logistic equation,

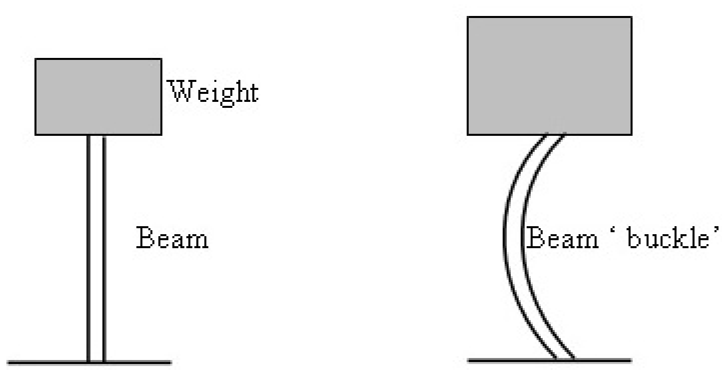
$$\dot{N} = rN(1 - N/K)$$

   using linear stability analysis.

Chapter 2

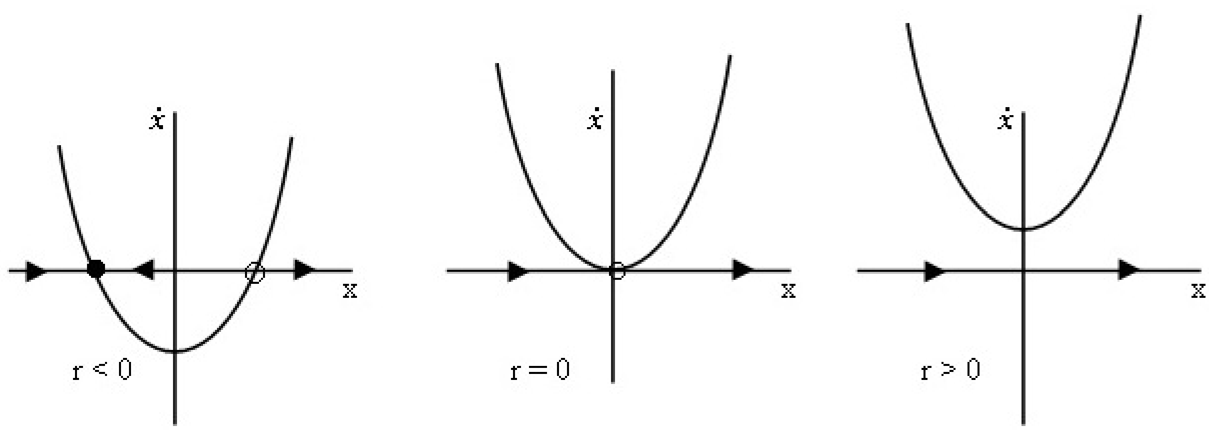
Bifurcations

All solutions of a dynamical systems either settle down to equilibrium or head out to $\pm\infty$. Fixed points can be created or destroyed, or their stability can change. Such qualitative change in the dynamics are called **bifurcations**, and the parameters values at which they occur are called **bifurcation points**. This is described in 1.7.1.
Scientifically, the bifurcations are important, as they provide models of transitions and instabilities when the control parameter is varied. Example:
Consider the buckling of a beam. If a small weight is placed on top of a beam the beam can support the load and remains vertical. But if the load is too heavy, the vertical position becomes unstable, and the beam may buckle. Here the weight plays, the role of the control parameter, and the deflection of the beam from vertical plays the role of the dynamical variable x.



2.1 Saddle-Node Bifurcation

Saddle node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

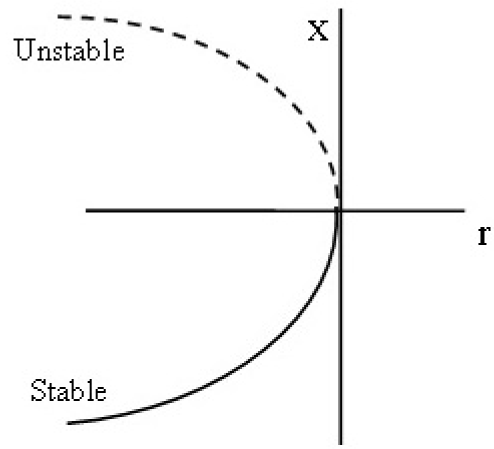


Example 2.1.1. *Consider the first order equation*

$$\dot{x} = r + x^2 \tag{2.1}$$

where r is a parameter, which may be positive, negative, or zero.

The figure shows the different cases, for $r < 0$ we have two fixed points, one stable and one unstable. As r approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half- stable fixed point at $x^* = 0$. This type of fixed point is extremely delicate-it vanishes as soon as $r > 0$ and now there are no fixed points at all.
In this example, we say that a *bifurcation* occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatrively different. The bifurcation diagram is shown below. $\dot{x} = 0$ gives the fixed points, for different valueds of r. Therefore, $x^2 = -r$.
The fixed points for $f(x) = x^2 + r$ are given by $x^* = \pm\sqrt{-r}$. To determine the linear stability, we compute $f'(x^*) = 2x^*$. Thus $x^* = -\sqrt{-r}$ is stable, since $f'(x^* < 0$. Similarly , $x^* = \sqrt{-r}$ is unstable. At the bifurcation point $r = 0$, we find $f'(x^*) = 0$, the linearization vanishes when the fixed point vanishes.



Theorem 2.1.1.

Suppose $x' = f_r(x)$ is a first order differential equation for which

1. $f_{r_c}(x^*) = 0$;
2. $f'_{r_c}(x^*) = 0$;
3. $f''_{r_c}(x^*) \neq 0$;
4. $\frac{\partial f_{r_c}}{\partial r}(x^*) \neq 0$.

Then this differential equation undergoes a saddle-node bifurcation at $r = r_c$

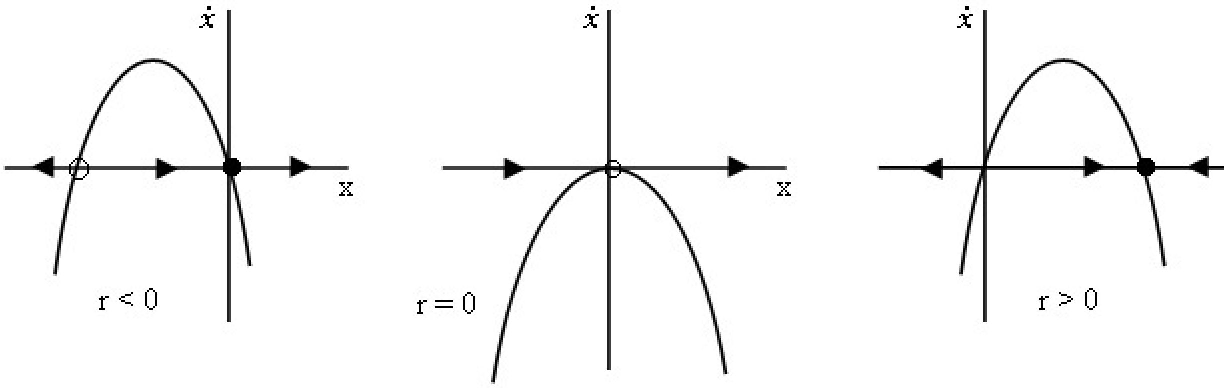
Exercises 2.

1. Give the linear stability analysis of the fixed point of $\dot{x} = r - 3x^2$ and give the bifurcation diagram.
2. Find the bifurcation points and the bifurcation diagram.

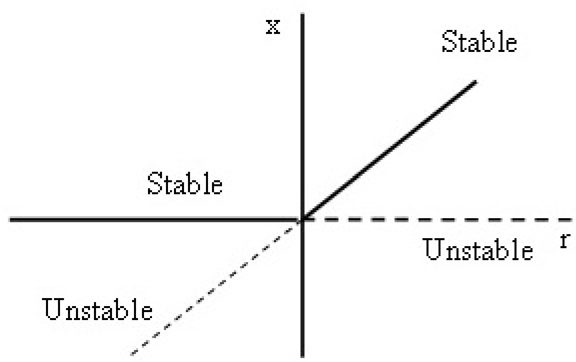
2.2 Transcritical Bifurcation

In the transcritical bifurcation, the two fixed points don't disappear after the bifurcation instead they just switch their stability.

Example 2.2.1. *Sketch the bifurcation diagram of* $\dot{x} = rx - x^2$.



For $r < 0$, there is an unstable fixed point at $x^* = r$ and stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r = 0$, the origin has become unstable, and $x^* = r$ is now stable. Figure shows the bifurcation diagram for the transcritical bifurcation, where the parameter r is regarded as the independent variable, and the fixed point $x^* = 0$ and $x^* = r$ are shown as dependent variables.



Bibliography

[1] Stevan H. Strogatz. *Nonlinear Dynamics and Chaos*, A Addition - Wesley Publishing Company, 1994.

[2] Morris W. Hirsch, Stephen Smale, Robert L. Devancy. *Differential Equations, Dynamical Systems and An Introduction to Chaos*, Acaademic Press, 2004.