Tribhuvan University
Institute of Science and Technology

Course Title: Mathematical Modeling	Full Marks: 50
Course No: Math 407	Pass Marks: 17.5
Level: B. Sc	Year: IV
Nature of course: Theory	Period per Week 4 Hrs

Unit 1: Modeling Change: 15 Lectures

Introduction, Mathematical models, Modeling change with difference equations, Approximating change with difference equations, Solution to dynamical systems, Systems of difference equations.

Unit 2: The Modeling Process, Proportionality and Geometric Similarity: 15 Lectures

Mathematical models, modeling using proportionality, modeling using geometric Similarity, Automobile gasoline mileage, Body weight and height, Strength and agility.

Unit 3: Model Fitting: 15 Lectures

Fitting model to data graphically, Analytical methods of data fitting, Applying the least squares criterion, Choosing a best model.

Unit 4: Optimization of Discrete Models: 15 Lectures

Introduction, An overview of optimization modeling, Linear programming, Geometric, Algebraic, Simplex method.

Unit 5: 15 Lectures

Population growth, Prescribing drug dosage, Braking distance revisited. Graphical Solutions of autonomous Differential equations.

Introduction

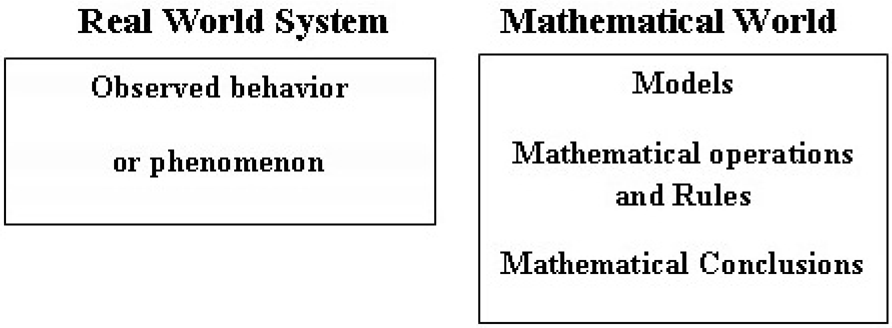
What is Modeling? Modeling of devices and phenomena is essential to both engineering and science.
So engineers and scientists have very practical reasons for doing mathematical modeling.

Real World: At the beginning, one has to identify **a real world (or external world)**
Conceptual World: a conceptual world (or mathematical world)

Introduction

- **Real World:**
  Various phenomena and behaviors, whether natural in origin or produced by artifacts, are observed.

- **Conceptual World:**
  The world of mind, which deals with the observation, modeling and prediction of the phenomena and behaviors which are happening in the real world.



Principles of Mathematical Modeling

1. Problem Identification
2. Making Assumptions; Variables Selection and their interrelationship
3. Solving and interpretation of the model
4. Verification of the model
5. Implementation of the model
6. Maintaining the model.

4, 5 and 6 steps are important for the stability of the solution of the modeled problem.

Where and when Mathematics can be used?

- There is no discipline free from mathematics. So we claim that mathematics is everywhere.

- Mathematics used to solve physical problems is called an applied mathematics.
- A physical problem means the problem in our surroundings.
  A stone thrown into the air,
  birds flying in the sky and fishes swimming in the water
  can be described by mathematics.

Where is Mathematics?

- Inside our room, we find cuboids beams, rectangular windows.
- The corners of the room can be considered as the origin and the surfaces are perpendicular to each other at the origin. One of the corners can be considered as the x- axis and other as the y-axis.
- We see the cracks in the field when there is no rain. The cracks in the ground are of different shapes which can be rectangular, cuboids, cubic, hexagonal or other irregular shapes.
  Also the trees on the sides of the road.

Example

- *Using trigonometry, the height of the tree can be approximated.*
  This idea is common in forest officers.
- *The volume and surface area of a human can be estimated crudely but quickly using the formula for the cylinder.*

For example;
If $r = 1/2$*ft. and* $h = 5$*ft. then,* $s = 2\pi rh$ *and* $v = \pi r2h$ *give* $s \approx 16ft2$ *and* $v \approx 4ft3$

Main Steps for Modeling

The method of describing the problems is called modeling. For the modeling we need to follow the following steps;

1. Problem and set up a model

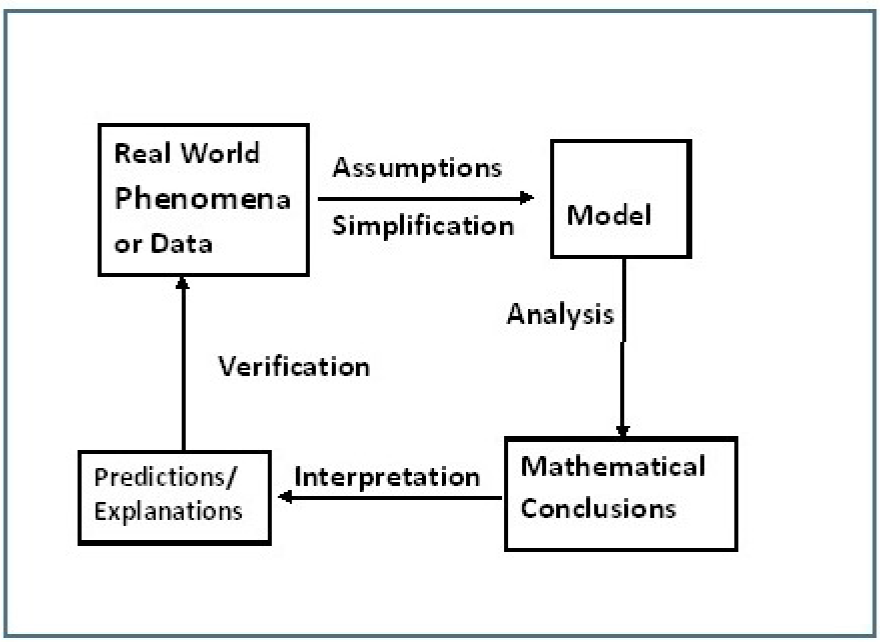
3. Stability validity of the solution.

Examples of Modeling
Example 1: The product of the age of a father and a son is 800. Find their ages if the son is 20 years younger than the father.
Solution: Setting a model: Let x is the age of the son then $x(x + 20) = 800$
Solving we get
$x = 20$ and $= -40$: Analysis of the problem.
$x \neq -40$,
$\therefore x = 20$: Interpretation and validations of the problem
Use of Mathematics in Biology

- **External Bio-fluid Dynamics:** People invented big airplanes seeing the birds flying in the sky.

Example: Eagles and other big birds
Small planes like helicopters were invented: seeing the small birds like humming birds.
Seeing the fish in the water: ships and marines were invented.

- **Internal Bio-fluid Dynamics:** Flow of blood, urine, cerebro-spinal and transport of mass and heat within an animal and flow of air in lungs.



Detailed Course:

Unit 1 Modeling Change

In modeling our world, we are often interested in predicting the value of a variable at sometime in the future. Examples are:

- population after some years
- real state value after some years
- the number of people with a communication disease after some time.
-

That is, mathematical model can help us to understand a behavior better or aid us in planning for the future. See the previous figure.

Models can only approximate real world behavior. One very powerful simplifying realationship is proportionality:

Definition: Two variables y and x are proportional (to each other) if one is always a constant multiple of the other- that is, if

$$y = kx$$

for some non zero constant k. We write $y \alpha x$

1.1 Modeling Change with Difference Equation

A powerful paradigm to use in modeling change is

future value = present value + change

By collecting the data over a period of time and plotting those data, we may capture the trend of the change in the behavior. If the behavior is talking place over discrete path periods then we are concerned with **difference equation**. If the behavior is taking place continuously with respect to time then we are concerned with the **differential equation.**

Definition: For a sequence of numbers A = $\{a_0, a_1, a_2, \ldots..\}$ the first differences are

$$\Delta a_0 = a_1 - a_0$$
$$\Delta a_1 = a_2 - a_1$$
$$\Delta a_2 = a_3 - a_2$$
$$\ldots$$
$$\Delta a_n = a_{n+1} - a_n$$

n a positive integer. **Example:** A Saving Certificate Consider the value of a savings certificate initially worth *Rs* 1000 that accumulates interest paid each month at 1 percent per month. Then

A = {1000, 1010, 1020.10, 1030.30, ……}

The first differences are

$$\Delta a_0 = a_1 - a_0 = 1010 - 1000 = 10$$
$$\Delta a_1 = a_2 - a_1 = 1020.20 - 1010 = 10.10$$
$$\Delta a_2 = a_3 - a_2 = 1030.30 - 1020.10 = 10.20$$

Example: A Saving Certificate (continuous)

Note that the first differences represent the damage in the sequence during one time period, or the interest. The first difference is useful for modeling change talking place in discrete intervals. Here, the change in the value of the certificate from one month to the next month is merely the interest paid during that month. If n is the number of months and an the value of the certificate after after n months, then the change or interest growth in each month is represented by the *nth* difference

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n$$

which can be rewritten as

$$a_{n+1} = a_n + 0.01a_n$$

Example: A Saving Certificate (continuous)

As we know that the initial deposit is Rs1000 we have the dynamical system model

$$a_{n+1} = 1.01a_n,\ n = 0, 1, 2..$$
$$a_0 = 1000$$

APPROXIMATING CHANGE WITH DIFFERENCE EQUATIONS

If *Rs* 50 is drawn from the account each month, the change during a period would be the interest earned during that period minus the monthly withdrawal. That is

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n - 50$$

Example: Mortagaging a Home , See the book.

Definitions

A **sequence** is a function whose domain is the set of all nonnegative integers and whose range is a subset of the real numbers.

A **dynamical system** is a relationship among terms in a sequence.

A **numerical solution** is a table of values satisfying the dynamical systems.

1.2 Approximating Change with Difference Equations

In most of the examples, describing the change mathematically is not precise. For such conditions, we must plot the change, observe a pattern, and then approximate the change in mathematical terms. Here we approximate some observed change to complete the expression

$$change = \Delta a_n = some\ function\ of\ f$$

Example: Growth of a Yeast Culture:

Data collected in a yeast culture experiment is given in the table. The graph represents the assumption that the change in population is proportional to the current size of the population. That is,

$$\Delta p_n = (p_{n+1} - p_n) = kp_n$$

where p_n represents the size of the population biomass after n hours, and k is a positive constant. The value of k depends on the time measurement.

Example 1: Growth of a Yeast Culture:

Model Refinement: Modeling Births, Deaths, and Resources:

If both births and deaths during a period are proportional to the population, then the change in population should be proportional to the population, as in the Example 1. However, certain resources (e.g., food) can support only a maximum population level rather than one that increases indefinitely. As these maximum levels are approached, growth should slow. These are described in the graphs.

Example 2: Growth of Yeast Culture Revisited

The data in the figure shows change in biomass beyond the eight observations. Here, in the third column of the data note that the change population per hour becomes smaller as the resources become more limited or constrained. We see the population appears to be approaching a limiting value, or **carrying capacity**, may guessed from figure as 665. As p_n approaches 665 the change is slow. Because 665 **-** p_n gets smaller as p_n approaches 665, we propose the model

$$\Delta p_n = p_{n+1} - p_n = k(665 - p_n)\, p_n$$

Example 2: Growth of Yeast Culture Revisited

The value of k can be estimated by 0.00082, giving the model

$$\Delta p_n = p_{n+1} - p_n = 0.00082(665 - p_n)\, p_n$$

Solving the model numerically for p_{n+1}, we have

$$p_{n+1} = p_n + 0.00082(665 - p_n)\, p_n$$

Model Refinement: Modeling Births, Deaths, and Resources:

Yeast biomass approaching a limiting population level

1.3 Solution to Dynamical Systems

Example: Saving Certificate Revisited

In the saving certificate example a savings certificate initially worth Rs1000 accumulated interest paid each month at 1percent of the balance. No deposit or withdrawals occured in the account, determining the dynamical system

$$a_{n+1} = 1.01a_n,\ a_0 = 1000 \qquad (1.1)$$

From the graph we see that the sequence $\{a_0, a_1, a_2, \ldots\}$ grows without bound. **Example: Saving Certificate Revisited**

Algebraically, we see the graph pattern

$$a_1 = 1010.00 = (1.1)a_0 = (1.1)1000 = 1.01(1000)$$

$$a_2 = 1020.10 = (1.01)a_1 = (1.01)(1.1)1000 = (1.1)^2 1000$$

$$a_3 = 1030.30 = (1.01)a_2 = (1.01)(1.1)^2 1000 = (1.01)^3(1000)$$

$$\ldots$$

The pattern of the sequence suggests that the *kth* term a_k is the amount 1000 multiplied by $(1.01)^k$. **Example:Saving Certificate Revisited**

Conjecture: For k = 1. 23, .. the term a_k in the dynamical system (1.1) is

$$(1.01)^k 1000 \qquad (1.2)$$

Test the Conjecture: We test the conjecture by examining whether the formula for a_k satisfies the system (1.1) upon substitution.

$$(1.01)^{n+1}1000 = (1.01)[(1.01)^{n}1000] = (1.01)^{n+1}1000$$

Example: Saving Certificate Revisited
Conclusion: The solution for the term a_k in the dynamical system (1.1) is

$$a_k = (1.01)^k 1000$$
$$\text{or } a_k = (1.01)^k a_0$$

which computes the balance a_k in the account after *k* months.
Linear Dynamical Systems $a_{n+1}ra_n$**, for** r **Constant** (see the book)
Example 2: Sewage Treatment Long Term Behavior of $a_{n+1} = ra_n$, for *r* Constant
Example 3: Prescription for Digoxin, Example 4: An Investment Annuity, Finding and Classifying Equilibrium Values
Theorem 2, 3, Example 6

1.4 System of Difference Equations

Example 1: Car Rental Company, Example 2: The Battle of Trafalgar **Exercises on Unit 1: 1.1:** 1,2,3, 4,5. **1.2:** 1, **1.3:** 1, 2, 3, **1.3:** 1

Unit 2: The Modeling Process, Proportionality and Geometric Similarity

2.1 Mathematical Models

Construction of Models: The following steps are important while constructing a model:
1. Identify the problem.
2. Make assumptions
   (a)Identify and classify the variable. (b)Determine interrelationships between the variables and submodels
3. Solve the model
4. Verify the model
   (a)Does it adress the problem? (b)Does it make common sense? (c) Test it wih real world data.
5. Implement the model.
6. Maintain the model.

2.2 Modeling Using Proportionality

Modeling Using Proportionality: Examples from the book. Modeling Vehicular Stopping Distance

2.3 Modeling Using Geometric Similarity

Modeling Using Geometric Similarity: Definition, Example 1: Raindrops from a Motionless cloud. Testing Geometric Similarity.
Example 2: Modeling a Bass Fishing Derby.

2.4 Automobile Gasoline Mileage

2.5 BodyWeight and Height, Strength and Agility

Unit 3 Model Fitting

3.0.1 Introduction

Introduction, Relationship Between Model Fitting and Interpolation,Sources of Error in the Modeling Process.

3.0.2 Fitting Models to Data Graphically

Visual Model Fitting with the original Data, Transforming the Data

3.1 Analytical Method of Model Fitting

Chebyshev Approximation Criterion, Least Square Criterion, Relating the Criteria

3.2 Applying the Least Square Criterion

Fitting a Straight Line, Fitting a power Curve

Unit 4 Optimization of Discrete Models

Introduction, Overview of Optimization Modeling. Example 1
Classifying Some Optimization Problems, Unconstrained Optimization Problems, Integer Optimization Programs: Examples 2, 3
Multiobjective Programming: An Investment Problem, Dynamic Programming Problems

4.1 Linear Programming 1: Geometric Solutyions
Interpreting a Linear Program Geometrically
Example 1: Carpenter's Problem, Example 2: Data Fitting Problem, Model Interpretation, Empty and Unbounded Feasible Regions, Level Curves of the Objective Function.
Theorem 1 (No Proof)

4.2 Linear Programming II: Algebraic Solutions
Linear Programming II: Algebraic Solutions. Example 1 Solving the Carpenter,s Problem Algebraically

4.3 Linear Programming III: The Simplex Method
Linear Programming III: The Simplex Method, Example 1 Carpenter,s Problem Revisited. Example 2 and Related Examples

Unit 5 Modeling using Differential Equations

Introduction: The Derivative as a Rate of Change, The Derivative as the Slope of the Tangent Line,

5.1 Population Growth
Malthus model: The equation $\dot{x} = ax$, $a > 0$ represents is a simplest model for the population growth, where x(t) measures the population of some species at time t. The equation tells that the growth rate of population is directly proportional to the size of the population. Such model does not consider the different circumstances like, famine, diseases, war etc, which are the bounds in the increment of the population. To describe the circumstances, Logistic model is used.

Example 5.1.1. *The Logistic Population Model*
The model considers the followings:

1. If the population is small, the growth rate is nearly directly proportional to the size of the population.

2. If the population is too large, the growth rate became negative.

The differential equation satisfying the assumptions is

$$\dot{x} = ax(1 - x/N), \qquad (5.1)$$

where a > 0 is a parameter, gives the rate of population growth when x is small, while N > 0 is a parameter, represents carrying capacity of the population. If x is small,

$$\dot{x} = ax$$

If x > N, $\dot{x} < 0$. Without loss of generality, let N = 1, then

$$\dot{x} = ax\,(1 - x) \qquad (5.2)$$

is a first order, autonomous, nonlinear equation. The solution of this equation is

$$x(t) = \frac{Ke^{at}}{1+Ke^{at}}$$

where K is determined at time t = 0 as

$$K = \frac{x(0)}{1-x(0)}$$

$$\therefore\ x\,(t) = \frac{x(0)e^{at}}{1-x(0)+x(0)e^{at}}$$

5.2 Prescribing Drug Dosage
Prescribing Drug Dosage

5.3 Braking Distance Revisited
%frametitleBraking Distance Revisited

5.4 Graphical Solution of Autonomous Differentia Equations
Equilibrium values or rest points, Examples 1, 2, 3

Text/Reference Books:

1. Frank R. Giordano, William P. Fox, Steven B. Horton, Maurice D. Weir; *Mathematical Modeling, Principles and Applications*, Cengage Learning, India Edition.
2. Sandip Banerjee; *Mathematical Modeling, Models, Analysis and Applications*, A Chapman and HallBoyce, W. and Book.
3. Boyce, W. and DiPrima, R.; *Elementary Differential Equations and Boundary Value Problems*, 9th Ed., Wiley India.

Guidelines to the question setter

There will be 5 questions each carrying 10 marks. All the questions are compulsory. There will be **two** OR choices in any question number from the same unit. The examination period of Math 407 will be 2 hours.

On the basis of the guidelines mentioned, we enclose one set of model question for Mathematical Modeling (Math 407)

MODEL QUESTION

Tribhuvan University

Bachelor Level / IV year/ Sc. & Tech. Full Marks: 50

Mathematical Modeling (Math 407) Time: 2 Hours

Candidates are required to give their answers in their own words as far as practicable. Attempt ALL the questions.

.

1. (a) Write out the first five terms a_0 - a_4 of $a_{n+1} = a^2_n$, $a_0 = 1$ [3]

(b) Write out the first four algebraic equations of $a_{n+1} = 3a_n$, $a_0 = 1$ for n = 0, 1, 2, 3. [3]

(c) If you currently have Rs 5000 in a saving account that pays 0.5 percent each month and you add Rs 200 each month, formulate a dynamical system. [4]

OR

(a) For the linear dynamical systems $a_{n+1} = ra_n$, for r constant, prove that $a_k = r^k a_0$, where a_0 is a given initial value. [5]

(b) A sewage treatment plant processes raw sewage to produce usable fertilizer and clean the water by removing all other contaminants. The process is such that each hour 12 percent of remaining contaminants in a processing tank are removed. What percentage of the sewage would remain after 1 day? How long would it take to lower the amount of sewage by half? How long until the level of sewage is down to 10 of the original level? [5]

2. (a) When two objects are geometrically similar? If two cuboids are considered, whose length breadth and height are l, b, h and l', b', h'. Prove that if y = f(l, S, V), where l, S, V represent lenght, surface area and volume , respectively, then $y = g(l, l^2, l^3)$ [5]

(b) Find the terminal velocity of a rain drop from a motionless cloud, assuming the constant gravity. [5]

3. Classify the describe the different types of errors. Fit $y = Ax^2$ to the data and predict the value at x = 2:25

x	0.5	1.0	1.5	2.0	2.5
y	0.7	3.4	7.2	12.4	20.1

[5 + 5]

4. Solve the linear program represented by the data given and the model y = c*x* with the largest absolute deviation ri = |yi - y(xi)|

x	1	2	3
y	2	5	8

[10]

5. (a) Give mathematical model for the population growth by Malthus. Explain why this model is not realistic. [5]

(b) Consider the model for the cooling of a hot cup of soup;

$$\frac{dT_m}{dt} = -k(Tm - \beta),\ k > 0$$

where Tm(0) = α. Here Tm is the temperature of the soup at any time t >0, β is the constant temperature of the surrounding medium, α is the initial temperature of the soup, k is a constant of proportionality depending on the thermal properties of the soup. Find Tm. [5]

OR

Apply the phase line techniques to obtain solution curves for the logistic growth equation $\frac{dP}{dx} = r(M - P)$ [10]