

# Dynamics of distal actions on locally compact groups

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Lectures Notes

## 1 Distal Maps

In this lecture we introduce distal maps (automorphisms) and discuss some properties.

Let  $X$  be a Hausdorff ( $T_2$ ) topological space. Let  $T : X \rightarrow X$  be a continuous bijective map.  $T$  is said to be distal for any two distinct points  $x, y \in X$ , the closure of the double orbit  $\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}$  in  $X \times X$  stays away from the diagonal, i.e.

$$\overline{\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}} \cap \{(a, a) \mid a \in X\} = \emptyset.$$

In particular,  $T$  is injective and if for any  $x \in X$  and any sequence  $\{n_k\} \in \mathbb{Z}$ , if  $T^{n_k}(x) \rightarrow a$  then  $T^{n_k}(y) \not\rightarrow a$  for any  $y \in Y$ .

Let us discuss some examples.

**Examples 1:** Let  $X = \mathbb{R}$ , then take an affine map  $T(x) = ax + b$ ,  $a, b \in \mathbb{R}$ . The topology on  $\mathbb{R}$  is the usual distance topology. Then  $T$  is distal if and only if  $|a| = 1$ . We show this for  $a > 0$ .

$$T^n(x) = a^n x + a^{n-1}b + \cdots + b = a^n x + \sum_0^{n-1} a^j b$$

Then  $\{T^n(x)\}$  is bounded if and only if  $a < 1$  and in this case  $T^n(x) \rightarrow b/(1-a)$  for all  $x$ .  $T^n(x) \rightarrow \infty$  if  $a \geq 1$ .  $\square$

**Example 2:** Let  $X = S^1$ , unit circle. Let  $T$  to be the rotation by an angle  $\alpha$ . i.e.  $Tx = cx$ , where  $c = e^{2\pi i\alpha}$ . Then  $T^n(x) = a^n x$  for every  $x$ . If  $T^{n_k}(x) \rightarrow a$  then  $a = c^l x$  where  $c^{n_k}(x) = c^l$  and  $T^{n_k}(y) \rightarrow c^l y$  for all  $y$ . Hence  $c^l x \neq c^l y$  unless  $x = y$ .  $\square$

**Example 3:** Let  $X = \mathbb{C}^d$ . Let  $T$  be a linear transformation in  $GL(n, \mathbb{C})$ . Then  $T$  is distal if and only if the eigenvalues of  $T$  are of absolute value 1.

Let  $\alpha$  be an eigenvalue of  $T$ . If  $|\alpha| < 1$ , then  $T^n(x) = \alpha^n(x) \rightarrow 0$  for some  $x \neq 0$  and  $T^n(0) = 0$ . So it is not distal. If  $|\alpha| > 1$  then  $T^{-n}x = \alpha^{-n}x \rightarrow 0$  for some  $x \neq 0$  and  $T^{-1}(0) = 0$ .  $\square$

Distal maps were first introduced by Hilbert and studied by many: Ellis, Furstenberg, Abels, Rossenblatt, Willis, Jaworski, Dani, Raja, Auslander etc.

We first have result by Ellis and Furstenberg independently on compact spaces. For a compact metric space  $X$  let  $X^X$  denote the set of all maps from  $X$  to  $X$  with the topology of point-wise convergence:  $f_n \rightarrow f$  in  $X^X$  iff  $f_n(x) \rightarrow f(x)$  for all  $x \in X$ . Here  $X^X$  is a compact space.

**Theorem 1.1** *If  $X$  is a compact space and  $T$  is a surjective continuous map. Then  $T$  is distal if and only if  $E(T) = \{T^n \mid n \in \mathbb{N}\}$  is a group.*

**Proof** Here,  $E(T)$  is clearly compact. We first show that  $E(T)$  is a semigroup,  $TE(T) \subset E(T)$ . (For, if take  $A \in E(T)$ , then  $A$  is a limit point of  $\{T^{n_k}\}$  in  $X^X$ . Then as  $T$  is continuous,  $TA$  is a limit point of  $\{T^{n_k+1}\}$ ). Hence  $T^n E(T) \subset E(T)$  for all  $n$ . We also have  $E(T)B \subset E(T)$  for  $B \in E(T)$ . (For, if  $A$  is a limit of  $\{T^{n_k}\}$ . Then  $AB$  is a limit point of  $T^{n_k}B \in E(T)$ ). Thus  $E(T)$  is semigroup. Let  $\mathcal{F}$  be the set of all closed subsemigroups

of  $E(T)$ .  $\mathcal{F}$  is nonempty as  $E(T) \in \mathcal{F}$ . If we put inclusion as an order of  $\mathcal{F}$ , then it has a minimal element say  $M$ . For  $J \in M$ , we have  $MJMJ = (MJM)A \subset MJ$ . Hence  $MJ = J$ . Hence there exists a  $A \in M$ , such that  $AJ = J$ . Take  $M' = \{C \in M \mid CJ = J\}$ . Then  $A \in M'$  and  $M'$  is also a semigroup and hence  $M' = M$ . Therefore,  $J \in M'$  and hence  $JJ = J$ , i.e.  $J$  is an idempotent. In particular since  $M$  is minimal,  $M = \{J\}$ , i.e. it consists of only one element which is an idempotent. If  $T$  is distal then  $T$  is one-one and hence a bijection. Suppose  $T$  is a limit point of  $\{T^{n_k}\}$  in  $X^X$ . Since  $T$  is distal  $T'$  is 1-1. Therefore, as  $J \in E(T)$  and for every  $x \in X$ ,  $J(x) = J^2(x) = J(J(x))$ , we have  $J(x) = x$ , i.e.  $J = I$  is an identity element. Similarly we show that for any  $S \in E(T)$ , as  $E(T)S$  is a closed (compact) semigroup, arguing as above, we have  $I \in E(T)S$ , therefore there exists  $S'$  such that  $S'S = S$  is invertible. I.e. every element in  $E(T)$  has a left inverse. Therefore,  $E(T)$  is a group. (Exercise: Show that in a semigroup with identity, if every element has a left inverse then every element is invertible and it is a group.)

Conversely, suppose  $T^{n_k}(x) \rightarrow c$  and  $T^{n_k}(y) \rightarrow c$  for some  $x, y, c \in X$  and an unbounded sequence  $\{n_k\}$ , then given a limit point  $S$  of  $\{T^{n_k}\}$  there exists a sequence  $\{m_k\}$  of  $\{n_k\}$  such that  $T^{m_k}(x) \rightarrow S(x) = c$  and  $T^{m_k}(y) \rightarrow S(y) = c$ . But as  $E(T)$  is a group,  $S$  is invertible and hence  $x = y$ . That is  $T$  is distal.  $\square$

**Corollary 1.2** *Let  $X$  be a compact Hausdorff space and  $T : X \rightarrow X$  be a bijective continuous map. If  $T$  is distal, then  $X$  is a disjoint union of minimal closed  $T$ -invariant sets, namely, closure of the orbits.*

**Proof** For  $x \in X$ , we have  $\overline{O_x} = \overline{\{T^n(x)\}_{n \in \mathbb{Z}}} = E(T)x$ , This is a minimal closed  $T$ -invariant set as  $T$  is distal. It is  $T$ -invariant since  $TE(T)x = E(T)x$ , as  $E(T)$  is a group. Moreover, for any  $x' \in E(T)x$ ,  $x' = Sx$  for some  $S \in E(T)$ . Then  $E(T)x'$  is the smallest closed  $T$ -invariant subset of  $E(T)x$  containing  $x'$ , but  $E(T)x' = E(T)Sx = E(T)x$  as  $E(T)$  is a group. Hence  $E(T)x$  is minimal. Now  $X$  is a disjoint union of  $E(T)x$  as for any pair of elements  $x, y \in X$ , either  $E(T)x = E(T)y$  or  $E(T)x \cap E(T)y = \emptyset$ .  $\square$

**Definition 1.3** : Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be bijective.  $T$  is said to be *equicontinuous* if the following holds: For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x, y \in X$  and  $d(x, y) < \delta$ , then  $d(T^n(x), T^n(y)) < \epsilon$ .

Any equicontinuous map is distal. Converse is not true even for a *minimal flow* (for any  $x \in X$ ,  $\{T^n(x)\}_{n \in \mathbb{Z}}$  is dense in  $X$ , i.e. there is no proper closed  $T$ -invariant set).

The following is well-known and can be found in Auslander's notes [2].

**Theorem 1.4** *On a compact (metric) space  $X$ , any minimal flow  $T$  is equicontinuous if and only if  $E(T)$  is a topological group and  $T$  is a translation (shift) by an element in the group.*

In particular, if  $T$  is minimal and equicontinuous and  $\{T^n\}_{n \in \mathbb{Z}}$  is dense in  $E(T)$ , which is a topological group,  $E(T)$  is abelian. Moreover, as  $T$  is minimal,  $X = E(T)x$  for every  $x \in X$ . Fix any  $x \in X$  and let  $H = \{S \in E(T) \mid S(x) = x\}$ , the stabiliser of  $x$  in  $E(T)$ . Then  $H$  is a closed subgroup of  $E(T)$ . Since  $E(T)$  is abelian, and  $X = E(T)x$ ,  $H$  stabilises every element of  $X$  and  $E(T)/H$  is an abelian group, moreover,  $X$  is isomorphic to  $E(T)/H$ .

**Corollary 1.5** *If a compact (metric) space admits a minimal equicontinuous flow if and only if it is an abelian topological group and  $T$  is a translation (shift) by an element.*

**Proof** Define a map  $\phi : E(T) \rightarrow X$  as  $\phi(S) = Sx$ . As  $E(T)$  is a topological group,  $\phi$  is continuous and onto. Since  $H$  stabilises every element of  $X$ , we have that there is a continuous map  $\bar{\phi} : E(T)/H \rightarrow X$   $\bar{\phi}(SH) = Sx$ . Now  $\bar{\phi}$  is a continuous bijection, and hence, a homeomorphism. Now  $T$  on  $X$  corresponds to a translation (shift) by  $T$  on  $E(T)/H$ .  $\square$

Have a look at Furstenberg's structure theorem for distal minimal transformation on compact metric spaces in [14]. It classifies all distal maps of a compact metric space. Any distal flow is a successive equicontinuous extensions starting from one point flow. It has also been generalised to non-metric spaces.

Now we are going to restrict ourselves to automorphisms of a topological group.

Let  $G$  be a locally compact Hausdorff topological group with the identity  $e$  and  $T \in \text{Aut}(G)$ . I.e.  $T$  is a bijective bi-continuous homomorphism. Then  $T$  is distal if and only if for every  $x \neq e$ ,

$$e \notin \overline{\{T^n(x) \mid n \in \mathbb{Z}\}}.$$

Throughout we will assume that our group is locally compact Hausdorff and second countable, (in particular, it is metrizable).

**Example 4:** Let  $G = \mathbb{T}^2 = S^1 \times S^1$ , 2-dimensional torus which is multiplicative group with the identity  $(1, 1)$ . Let  $T(t_1, t_2) = (t_1 t_2, t_2)$ . It is easy to see that  $T$  is distal. Here,  $T^n(t_1, t_2) = (t_1 t_2^n, t_2)$ . so if  $T^{n_k}(t_1, t_2) = (t_1 t_2^{n_k}, t_2) \rightarrow (e, e)$ , where  $e = (1, 0)$ . Then  $t_2 = e$  and hence  $t_1 = e$ .  $\square$

**Example 5:** Suppose  $G$  is a compact (non-abelian) group. Let  $k \in G$  and take  $T$  to be the inner automorphism by  $k$ . I.e.  $T(g) = k g k^{-1}$ . Then  $T$  is distal, in fact  $T^n(g) = k^n g k^{-n}$  for all  $n \in \mathbb{Z}$ . Then its limits points are of the form  $x g x^{-1}$  where  $x$  is a limit point of  $\{k^n \mid n \in \mathbb{Z}\}$ . So  $x g x^{-1} = e$  if and only if  $g = e$ . More generally, if  $K$  is a subgroup of  $G$  and if  $T$  is an inner automorphism of  $G$  by an element of  $k$  then  $T$  is distal.  $\square$

**Example 6:** Inner automorphisms of a nilpotent group (for e.g.  $G$  is the group of strictly upper-triangular matrices).

Let us define the *contraction group* of  $T$ ,  $C(T) = \{x \in G \mid T^n(x) \rightarrow e\}$ . If  $T$  is distal then by definition,  $C(T) = \{e\} = C(T^{-1})$ . What about the converse?

This is easily true for  $\mathbb{R}^d$ . As  $T \in GL(d, \mathbb{R})$  and  $C(T)$  is a  $T$ -invariant subspace and the eigenvalues of  $T$  restricted to  $C(T)$  are of absolute value less than 1 unless  $C(T)$  is trivial. We will discuss this later.

The first question we discuss is whether distality carries over if you go to the quotients modulo a compact invariant subgroup.

**Theorem 1.6** *Suppose  $K$  is a compact normal  $T$ -invariant subgroup of  $G$  for some  $T \in \text{Aut}(G)$ . Let  $\bar{T} \in \text{Aut}(G/K)$  be the corresponding map defined as  $\bar{T}(xK) = T(x)K$ , for  $x \in G$ . Then the following holds:  $T$  is distal on  $G$  if and only if  $\bar{T}$  is distal and  $T|_K$ , the restriction  $T$  to  $K$ , is also distal.*

**Proof** Suppose  $\bar{T}$  is distal and  $T|_K$  is distal. For some  $x \in G$ , suppose  $T^{n_k}(x) \rightarrow e$  for some sequence  $\{n_k\}$ . Then  $T^{n_k}(x)K \rightarrow eK = K$ . Hence  $\bar{T}^{n_k}(xK) \rightarrow K$  in  $G/K$ . Since  $\bar{T}$  is distal, it implies that  $xK = K$  i.e.  $x \in K$ . Since  $T|_K$  is distal, we have  $x = e$ . Hence  $T$  is distal.

Conversely, assume that  $T$  is distal on  $G$ . Then its restriction to  $K$  is obviously distal. We show that  $\bar{T}$  is distal. Suppose  $(\bar{T})^{n_k}(xK) = T^{n_k}(x)K \rightarrow K$  for some  $x \in G$ . We have to show that  $xK = K$ , i.e.  $x \in K$ . This implies

that  $T^{n_k}(x) = u_k h_k$  for some  $u_k \rightarrow e$  and  $h_k \in K$ . Since  $K$  is compact, passing to a subsequence if necessary, we may assume that  $h_k \rightarrow h \in K$  and hence  $T^{n_k}(x) \rightarrow h$ . Since  $T|_K$ ,  $E(T|_K)$  is a group in  $K^K$  (by Theorem 1.1). Let  $\gamma \in E(T|_K)$  be the limit point of  $\{(T|_K)^{n_k}\}$ . Let  $b = \gamma^{-1}(h) \in K$ . Passing to a subsequence if necessary, we may assume that  $T^{n_k}(b) \rightarrow \gamma(b) = h$ . Since  $T$  is distal on  $G$ ,  $x = b \in K$ , i.e.  $xK = K$  and  $\bar{T}$  is distal.  $\square$

The above theorem allows us to go to quotient by compact normal invariant subgroups. Any connected group has a maximal compact normal (characteristic) subgroup such that  $G/K$  is a (connected) Lie group.

Let  $G^0$  be the connected component of the identity  $e$ , i.e. it is the maximal connected subset of  $G$  containing  $e$ . Then  $G^0$  is a closed normal subgroup of  $G$ .

A topological space is said to be *totally disconnected* if its connected components are singleton sets. For a topological group  $G$ , it is totally connected iff  $G^0 = \{e\}$ . In particular for a group  $G$ , the quotient group  $G/G^0$  is totally disconnected.

Examples of totally disconnected groups: Any group with discrete topology, finite groups,  $\mathbb{Z}$ ,  $GL(n, \mathbb{Z})$ , for a prime  $p$ , the field of  $p$ -adic numbers  $\mathbb{Q}_p$  with addition as operation, closed subgroups of  $GL_n(\mathbb{Q}_p)$ . Any (closed) subgroup of a totally disconnected group is totally disconnected.

If  $G$  is a totally disconnected locally compact group, then it has a neighbourhood bases of the identity  $e$  consisting of compact open subgroups.

The following by Jaworski and Raja characterises distal actions on totally connected groups.

**Theorem 1.7** *Let  $G$  be a totally disconnected locally compact group and let  $T \in \text{Aut}(G)$ . Then the following are equivalent:*

1.  $T$  is distal
2.  $C(T) = C(T^{-1}) = \{e\}$ .
3.  $G$  has a neighbourhood bases of  $T$ -invariant compact open subgroups.

Our next question is if distality is preserved if we go modulo the connected component of the identity  $e$ .

**Theorem 1.8** *For  $T \in \text{Aut}(G)$ , let  $\bar{T} : G/G^0 \rightarrow G/G^0$  is the automorphism defined from  $T$ , i.e.  $\bar{T}(xG^0) = T(x)G^0$  for all  $x \in G$ . Then  $T$  is distal if and only if  $T|_{G^0}$  is distal and the  $\bar{T}$  on  $G/G^0$  is distal.*

Proof: The ‘if’ part follows as in the proof of the previous theorem. Now we prove the ‘only if’ part. Suppose  $T$  is distal on  $G$ . Then its restriction to  $G^0$  is distal. We want to prove the distality of  $\bar{T}$ . We will show that  $C(\bar{T})$  is trivial. Suppose  $\bar{T}^n(xG^0) \rightarrow G^0$  in  $G/G^0$ . From Theorem 1.6, we can assume going modulo the maximal compact normal subgroup of  $G^0$ , that  $G^0$  is a group without any non-trivial compact normal subgroup. From the structure theory of locally compact groups, we can find a compact totally disconnected subgroup  $K$  of  $G$  such that  $K$  centralises  $G^0$  and  $G^0 \times K = H$  is open in  $G$ . Since  $T^n(x)G^0 \rightarrow G^0$ ,  $T^n(x) \in H$  for large  $n$ , and hence replacing  $x$  by  $T^n(x)$ , we may assume that  $x \in K \times G^0$  and  $T^n(x) \in K \times G^0$ . Let  $x = kg = gk$ , where  $k \in K$  and  $g \in G^0$ . Then  $T^n(k)G^0 \rightarrow G^0$ . Since  $K$  is totally disconnected and it centralises  $G^0$  and  $G^0$  is a group without any non-trivial compact normal subgroup we have that  $T^n(k) \in K$  and moreover, as  $T^n(k)G^0 \rightarrow G^0$ , we can show that  $T^n(k) \rightarrow e$  and hence  $k = e$ . Then  $x = g \in G^0$ . I.e.  $C(\bar{T})$  is trivial. Similarly,  $C(\bar{T}^{-1})$  is trivial and hence  $\bar{T}$  is distal from Theorem 1.7.

## 2 Distal Groups

In this section, we define distal and pointwise distal groups and discuss their properties: Let  $G$  be a locally compact (second countable) group.

$G$  is said to be *pointwise distal* if the inner automorphism by every element in  $G$  is distal. Equivalently, for every  $g \in G$ , and for every  $x \neq e$ .

$$e \notin \overline{\{g^n x g^{-n} \mid n \in \mathbb{Z}\}}.$$

$G$  is said to be *distal* if the conjugacy action of  $G$  on  $G$  is distal. Equivalently, for every  $x \neq e$ ,

$$e \notin \overline{\{g x g^{-1} \mid g \in G\}}.$$

**Example 7:** Abelian groups, here the conjugacy action of each element is the identity map. More generally, nilpotent groups.

**Example 8:** Compact groups:  $\{g x g^{-1} \mid g \in G\}$  is closed (and compact) and it doesn’t contain the identity  $e$  unless  $x = e$ .

**Example 9:** Discrete groups: If  $g_n x g_n^{-1} \rightarrow e$ , then  $g_n x g_n^{-1} = e$  for large  $n$  and hence  $x = e$ .

**Example 10:** If  $G$  is a compact extension of a nilpotent (normal) subgroup. I.e. if  $G$  has a closed (normal) nilpotent subgroup  $N$  such that  $G/N$  is compact. More generally,

**Lemma 2.1** *If  $H$  is a closed subgroup of  $G$  such that  $H$  is distal and  $G/H$  is compact then  $G$  is distal.*

**Corollary 2.2** *A group  $G$  is (pointwise) distal if and only if for any compact normal subgroup  $K$ ,  $G/K$  is (pointwise distal) and the conjugacy action of  $G$  on  $K$  is (pointwise) distal.*

**Corollary 2.3** *A group  $G$  is (pointwise) distal if and only if  $G/G^0$  is (pointwise) distal and for every  $g \in G$ , the conjugacy action of  $g$  restricted to  $G^0$  is distal.*

Subgroups of distal groups are distal. What about quotients: It is true from above corollaries that quotients of distal groups are distal if they are modulo compact group or connected components. (Or if the quotient is compact or discrete or nilpotent, it is anyway distal).

Quotients modulo closed normal subgroup of pointwise distal groups are pointwise distal. We do not know if the corresponding statement for the quotients of distal groups, except in some obvious cases.

Distal groups are obviously pointwise distal but the converse is not true as is shown by the following example which is well-known:

**Example 11:** Let  $G = S \ltimes K$ , a semidirect product of groups, where  $S = \cup_{n=1}^{\infty} S_n$ ,  $S_n$  is the group of all permutations of order  $n$ , and  $K = \prod_{i=1}^{\infty} K_i$ ,  $K_i = C$ , a nontrivial compact group for all  $i$ , for e.g.  $C = S^1$ . Here,  $S$  is endowed with discrete topology and  $K$ , being a product of compact groups, is compact. The action of  $S$  on  $K$  is given as follows: If  $\alpha$  is a permutation of order  $n$ , then

$$\alpha(k_1, \dots, k_n, k_{n+1}, \dots) \alpha^{-1} = (k_{\alpha(1)}, \dots, k_{\alpha(n)}, k_{n+1}, \dots).$$

Since  $G/K$  is discrete, it is enough to consider the conjugacy action of  $G$  on  $K$ . Here,  $K$  is distal, any element  $\alpha$  in  $S$  has finite order as an automorphism of  $K$ . Hence it acts distally on  $K$ . But if you take an element  $k = (a, e, \dots)$ ,  $a \neq e$  in  $C$ , and take  $\alpha_n = (1, n)$ , then  $\alpha_n k \alpha_n^{-1} = c_n$  whose all entries are  $e$  except for the  $n$ th entry which is  $a$ . So we have a non-trivial element  $k$  whose orbit closure contains the identity  $e$ .  $\square$



### 3 Concentration functions of a probability measure and point-wise distal groups

In this section, we are going to relate pointwise distal groups and behaviour of convolution powers of a probability measures; namely, the behaviour of concentration functions of a probability measure.

Let  $G$  be locally compact group. Let  $P(G)$  denote the set of all (regular Borel) probability measures on  $G$ . Then  $P(G)$  has a semigroup structure give by the convolution operation ‘\*’. If  $\mu, \lambda \in P(G)$  are two probability measures then  $\mu * \lambda = o(\mu \times \lambda)$  where  $o : G \times G \rightarrow G$  is the group operation  $o(x, y) = xy$ . i.e.

$$\mu * \lambda(B) = \mu \times \lambda(\{(x, y) \mid xy \in B\}) = \int_G \mu(By^{-1}) d\lambda(y) = \int_G \lambda(x^{-1}B) d\mu(x).$$

On  $\mathbb{R}$  for e.g. we will have

$$\mu * \lambda(B) = \mu \times \lambda(\{(x, y) \mid x + y \in B\}).$$

$P(G)$  is semigroup with the identity  $\delta_e$ , the Dirac measure at the identity  $e$  of the group, where  $\delta_e$  is defined as  $\delta_e(B) = 1$  if  $e \in B$  otherwise it is zero for any Borel set  $B$ .  $P(G)$  is a topological semigroup with weak topology, it is the smallest topology on  $P(G)$ , which makes  $\mu_\alpha \rightarrow \mu$ , if and only if  $\mu_\alpha(f) \rightarrow \mu(f)$  for all bounded continuous real valued functions  $f$  on  $G$ .

One can embed  $G$  in  $P(G)$  as  $g \mapsto \delta_g$ , the Dirac measure at  $g$ , and  $\delta_g * \delta_h = \delta_{gh}$ , for  $g, h \in G$ .

For any  $\mu \in P(G)$ , we define  $\mu^n = \mu * \dots * \mu$  ( $n$ -times).

Now we define *concentration functions* of a measure.

**Definition 3.1** For  $\mu \in P(G)$ , given a compact set  $K$  in  $G$  and  $n \in \mathbb{N}$ , the  $n$ th concentration function of  $\mu$  is

$$f_n(\mu, K) = \sup_{x \in G} \mu^n(Kx^{-1}).$$

Naturally, since  $\mu^n \in P(G)$ ,  $f_n(\mu, K)$  is bounded below by 0 and bounded above by 1.

Moreover,  $\{f_n(\mu, K)\}$  is a non-increasing (monotone) sequence for a given compact set  $K$ . For any  $x \in G$ ,

$$\mu^{n+1}(Kx^{-1}) = \mu^n * \mu(Kx^{-1}) = \int_G \mu(Kx^{-1}y^{-1}) d\mu(y) \leq \int_G f_n(\mu, K) d\mu(y) = f_n(\mu, K).$$

Since the above is true for any  $x \in G$ , for all  $n \in \mathbb{N}$ ,

$$f_{n+1}(\mu, K) = \sup_{x \in G} \mu^{n+1}(Kx^{-1}) \leq f_n(\mu, K).$$

We say that the concentration functions of  $\mu$  converge to zero if for every compact set  $K$  in  $G$ ,  $f_n(\mu, K) \rightarrow 0$ . Otherwise, we say that they do not converge to zero. i.e. there exists a compact set  $K$  for which it does not converge to zero.

Let us see some examples.

**Example 12:** Let  $G$  be any group (for e.g.  $G = \mathbb{R}$  or  $\mathbb{R}^d$ ) and  $\mu = \delta_g$  for some  $g \in G$ . Then  $\mu^n = \delta_{ng}$ . Then for any compact set  $K$ , for  $x = g^{-n}k$  for some  $k \in K$ , we have  $g^n \in Kx^{-1}$  and hence

$$\mu^n(Kx^{-1}) = \delta_{g^n}(Kx^{-1}) = 1.$$

Hence  $f_n(\mu, K) = 1$  for all  $n$ . The concentration functions of  $\mu$  do not converge to zero.

**Example 13:** Let  $G$  be a group,  $H$  be a compact subgroup and let  $g \in G$  be such that  $gHg^{-1} = H$ . Let  $\mu = \delta_g * \omega_H (= \omega_H * \delta_g)$ . Then  $\mu^n = \delta_{g^n} * \omega_H$  and  $f_n(\mu, H) = 1$  for all  $n$ .

**Example 14:** Let  $G$  be a compact group. Then  $P(G)$  is compact and the sequence  $\{\mu^n\}$  is relatively compact, hence for every  $\epsilon > 0$ , there exists a compact set  $K_\epsilon$  such that  $\mu^n(K_\epsilon) > 1 - \epsilon$ . Then  $f_n(\mu, K_\epsilon) > 1 - \epsilon$ .

**Example 15:** Let  $G$  be a discrete group. Then for any measure  $\mu$ ,  $\{f_n(\mu, K)\}$  converge to zero unless  $\mu = \delta_g * \lambda$  where  $\lambda$  is supported on a compact subgroup normalised by  $g$ .

**Example 16:** Let  $G = \mathbb{R}$ , suppose  $\mu$  be such that the support of  $\mu$  has at least two elements in it, for e.g.  $\mu = (\delta_0 + \delta_1)/2$ . Then the concentration functions of  $\mu$  do not converge to zero.

If the concentration functions of  $\mu$  do not converge to zero, then there exists an element  $x \in G$ , such that  $\{\mu^n * \delta_x^{-n}\}$  is relatively compact in  $P(G)$ .

In Example 2,  $\mu^n * \delta_{g^{-n}} = \omega_H$ .

The following result gives a structure of the group generated by the support of  $\mu$  in case the concentration functions of  $\mu$  do not converge to zero. It was obtained jointly with S.G. Dani for Lie groups and it is due to Jaworski, Rosenblatt and Willis for all locally compact groups.

**Theorem 3.2** *Let  $G$  be a locally compact group and  $\mu \in P(G)$  be such that the concentration functions of  $\mu$  do not converge to zero. Then there exists subgroups  $H \subset M$ ,  $H$  compact and an element  $x \in \text{supp}\mu$  which normalises both  $H$  and  $M$  such that  $\text{supp}\mu \subset xM = Mx$  and the conjugacy action of  $x$  on  $M/H$  contracts  $M/H$ . i.e. for all  $g \in M$ ,  $x^n g x^{-n} H \rightarrow H$  in  $G/M$ .*

This in particular, gives the structure of  $G(\mu)$ , the closed subgroup generated by the support of  $\mu$ .  $G(\mu) = \mathbb{Z} \ltimes M$ ,  $G(\mu)/H = \mathbb{Z} \ltimes M/H$  and for  $x = 1$ , the generating element of  $Z$ , the conjugacy action of  $x$  on  $M/H$  contracts whole of it.

We say that a group  $G$  has shifted convolution property if given any  $\mu \in P(G)$ , either the concentration functions of  $\mu$  converges to zero or there exist a compact subgroup  $H$  and an element  $x \in G$  which normalises  $H$  such that  $\mu^n * \delta_{x^{-n}} \rightarrow \omega_H$ .

Note that in the second case, we have that  $\text{supp}\mu \subset xH = Hx$  and hence for every  $g \in \text{supp}\mu$ ,  $\mu^n * \delta_{g^{-n}} \rightarrow \omega_H$ .

The following is joint work with C.R.E. Raja.

**Theorem 3.3** *A locally compact group is point-wise distal if and only if it has shifted convolution property.*

Idea of Proof for the ‘if’ part: Suppose  $G$  is pointwise distal. Let  $\mu \in P(G)$  be such that the concentration functions of  $\mu$  do not converge to zero. Then  $G(\mu)$ , the closed subgroup generated by the support of  $\mu$  is also pointwise distal. Then from Theorem 3.2, it has a compact normal subgroup  $H$  and  $G(\mu)/H = \mathbb{Z} \ltimes M/H$  and the conjugacy action of  $x = 1$  in  $\mathbb{Z}$  contracts  $M/H$ . If  $H \neq M$ ,  $G(\mu)/H$  is not pointwise distal, As  $H$  is compact, by Corollary 2.2,  $G(\mu)$  itself is not pointwise distal, a contradiction. Hence,  $H = M$  is compact. Hence  $\mu = \delta_x * \lambda$ , where  $\lambda$  is supported on the compact group  $H$ . Now since  $G(\mu) = \mathbb{Z} \ltimes H$  is pointwise distal, one can show that  $\mu^n x^{-n} \rightarrow \omega_H$ .

## 4 Distality and Minimal Orbit Closures

For a automorphism  $T$ , and  $x \in G$ , we call  $O_x = \{T^n(x) \mid n \in \mathbb{Z}\}$ , the orbit of  $x$ . A map  $T$  on a group  $G$  is said to have the property [MOC] if for every  $x \in X$ , the orbit closure  $\overline{\{T^n(x) : n \in \mathbb{Z}\}}$  is a minimal closed  $T$ -invariant set. I.e. there is no proper closed  $T$ -invariant subset of  $C_x = \overline{O_x} = \overline{\{T^n(x)\}}$ . This is equivalent to the following: For any  $x, y \in G$ , if  $y \in C_x$ , then  $C_y = C_x$

(i.e.  $x \in C_y$ ); In other words, if  $T^{n_k}(x) \rightarrow y$  for some sequence  $\{n_k\}$ , then  $T^{m_k}(y) \rightarrow x$  for some sequence  $\{m_k\}$ ).

[MOC] stands for ‘*minimal orbit closures*’.

If  $T$  has [MOC], then  $G$  splits into disjoint minimal closed  $T$  invariant sets.

If  $T$  has [MOC], then  $T$  is distal. (For if  $T^{n_k}(x) \rightarrow e$  for some  $x \neq e$ . Then  $C_x$  is not minimal as  $\{e\}$  is a proper closed  $T$ -invariant subset of  $C_x$ .)

The above holds also for an action of any subgroup of  $\text{Aut}(G)$ .

What about the converse?

For a compact group, distality and [MOC] are equivalent, (see Corollary 1.2).

If  $G$  is a discrete group, then any set is closed and hence the orbit of  $x$ ,  $O_x = C_x$ . Therefore, if  $y \in C_x$ , then  $y = T^m(x)$  for some  $m$  and hence  $x = T^{-m}(y) \in O_y = C_y$ , hence,  $C_x$  is a minimal closed  $T$ -invariant set.

Let  $G = \mathbb{R}^d$  and  $T \in \text{Aut}(G)$  be distal. Suppose  $T^{n_k}(x) \rightarrow y$  for some  $x, y$ . Take  $W = \{w \in \mathbb{R}^d \mid T^{n_k}(w) \text{ converges}\}$ . Then  $W$  is a closed  $T$ -invariant subspace and  $x \in W$ . Since  $W$  is  $T$ -invariant and closed, it contains  $T^n(x)$  for all  $n$  and also  $y$ . So  $W$  is non-trivial. Moreover, the restriction of  $T^{n_k}$  to  $W$  converges point-wise, we have that  $T^{n_k}|_W$  converges to (say)  $\psi$ , which is a homomorphism. Since  $T$  is distal  $\psi$  is injective and hence  $\psi \in \text{Aut}(W) = GL(W)$ . Then  $y = \psi(x)$  and  $T^{-n_k}|_W \rightarrow \psi^{-1}$ . Hence  $T^{-n_k}(y) \rightarrow \psi^{-1}(y) = x$ .

On a connected Lie group, distality and [MOC] are equivalent (cf. Abels). In fact, Abels proved it for an action of any automorphism group.

If  $G$  is totally disconnected then  $T$  is distal implies  $T$  has [MOC] (this follows from Theorem 2.3. This is because,  $G$  has a neighbourhood bases of the identity  $e$  consisting of  $T$ -invariant compact open subgroups say  $K_\alpha$ . Then  $G/K_\alpha$  is a discrete space and the corresponding maps  $T_\alpha : G/K_\alpha \rightarrow G/K_\alpha$  is distal and also has [MOC] as the orbits are closed. This implies that  $C_x K_\alpha = O_x K_\alpha$  for all  $x \in G$  and every  $K_\alpha$ . if  $x \in G$ , and  $y \in C_x$ , then  $C_x K_\alpha = C_y K_\alpha$  for all  $K_\alpha$ . Since  $\bigcap_\alpha K_\alpha = \{e\}$ . We get that  $C_x = C_y$ .

**Theorem 4.1** *Let  $G$  be locally compact group and  $T \in \text{Aut}(G)$ . Let  $K$  be a compact (normal)  $T$ -invariant subgroup of  $G$ . Then  $T$  has [MOC] if and only if  $T|_K$  has [MOC] and  $\bar{T}$  on  $G/K$  has [MOC].*

**Theorem 4.2** *Let  $G$  be a locally compact group and  $T \in \text{Aut}(G)$ . Then  $T$  is distal if and only if  $T$  has [MOC].*

The above is also true for the action of a locally compact compactly generated abelian group of automorphisms.

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