Dynamics of distal actions on locally compact groups

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Lectures Notes

1 Distal Maps

In these lectures we introduce distal maps and distal group automorphisms and discuss some properties.

Let X be a Hausdorff (T_2) topological space. Let $T : X \to X$ be a continuous bijective map. T is said to be distal for any two distinct points $x, y \in X$, the closure of the double orbit $\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}$ in $X \times X$ stays away from the diagonal, i.e.

$$\overline{\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}} \cap \{(a, a) \mid a \in X\} = \emptyset.$$

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In particular, T is injective and if for any $x \in X$ and any sequence $\{n_k\} \in \mathbb{Z}$, if $T^{n_k}(x) \to a$ then $T^{n_k}(y) \not\to a$ for any $y \in Y$.

Let us discuss some examples.

Examples 1: Let $X = \mathbb{R}$, then take an affine map $T(x) = ax + b$, $a, b \in \mathbb{R}$. The topology on $\mathbb{R}$ is the usual distance topology. Then T is distal if and only if $|a| = 1$. We show this for $a > 0$.

$$T^n(x) = a^n x + a^{n-1}b + \cdots + b = a^n x + \sum_{0}^{n-1} a^j b$$

Then $\{T^n(x)\}$ is bounded if and only if $a < 1$ and in this case $T^n(x) \to b/(1-a)$ for all x. $T^n(x) \to \infty$ if $a \geq 1$. □

Example 2: Let $X = S^1$, unit circle. Let T to be the rotation by an angle α. i.e. $Tx = cx$, where $c = e^{2\pi i\alpha}$. Then $T^n(x) = a^n x$ for every x. If $T^{n_k}(x) \to c^{n_k}x$ converges to a then $a = c'x$ where $c^{n_k}(x) = c'$ and $T^{n_k}(y) \to c'y$ for all y. Hence $c'x \neq c'y$ unless $x = y$. □

Example 3: Let $X = \mathbb{C}^d$. Let T be a linear transformation in $GL(n, \mathbb{C})$. Then T is distal if and only if the eigenvalues of T are of absolute value 1. Let α be an eigenvalue of T. If $|\alpha| < 1$, then $T^n(x) = \alpha^n(x) \to 0$ for some $x \neq 0$ and $T^n(0) = 0$. So it is not distal. If $|\alpha| > 1$ then $T^{-n}x = \alpha^{-n}x \to 0$ for some $x \neq 0$ and $T^{-1}(0) = 0$. □

Distal maps were first introduced by Hilbert and studied by many: Ellis, Furstenberg, Abels, Rossenblatt, Willis, Jaworski, Dani, Raja, Auslander etc.

We first have result by Ellis and Furstenberg independently on compact spaces. For a compact metric space X let X^X denote the set of all maps from X to X with the topology of point-wise convergence: $f_n \to f$ in X^X iff $f_n(x) \to f(x)$ for all $x \in X$. Here X^X is a compact space.

The following theorem follows from Ellis [11].

Theorem 1.1 *If X is a compact space and T is a surjective continuous map. Then T is distal if and only if $E(T) = \overline{\{T^n \mid n \in \mathbb{N}\}}$ is a group.*

Proof Here, $E(T)$ is clearly compact. We first show that $E(T)$ is a semigroup, $TE(T) \subset E(T)$. (For, if $A \in E(T)$, then A is a limit point of $\{T^{n_k}\}$ in X^X. Then as T is continuous, TA is a limit point of $\{T^{n_k+1}\}$). Hence $T^nE(T) \subset E(T)$ for all $n \in \mathbb{N}$. We also have $E(T)B \subset E(T)$ for $B \in E(T)$. (For, if A is a limit of $\{T^{n_k}\}$. Then AB is a limit point of

$T^{n_k}B \in E(T)$). Thus $E(T)$ is semigroup. Let $\mathcal{F}$ be the set of all closed subsemigroups of $E(T)$. $\mathcal{F}$ is nonempty as $E(T) \in \mathcal{F}$. If we put inclusion as an order of $\mathcal{F}$, then it has a minimal element say M. For $J \in M$, we have $MJMJ = (MJM)A \subset MJ$. Therefore, MJ is a closed semigroup contained in M and as M is minimal, $MJ = M$. Hence there exists a $A \in M$, such that $AJ = J$. Take $M' = \{C \in M \mid CJ = J\}$. Then $A \in M'$ and M' is also a closed semigroup contained in M and hence $M' = M$. Therefore, $J \in M'$ and hence $JJ = J$, i.e. J is an idempotent. In particular since M is minimal, $M = \{J\}$, i.e. it consists of only one element which is an idempotent. If T is distal then T is one-one and hence a bijection. Suppose T' is a limit point of $\{T^{n_k}\}$ in X^X. Since T is distal T' is 1-1. Therefore, as $J \in E(T)$ and for every $x \in X$, $J(x) = J^2(x) = J(J(x))$, we have $J(x) = x$, i.e. $J = I$ is an identity element. Similarly we show that for any $S \in E(T)$, as $E(T)S$ is a closed (compact) semigroup, arguing as above, we have $I \in E(T)S$, therefore there exists S' such that $S'S = S$. I.e. every element in $E(T)$ has a left inverse. Therefore, $E(T)$ is a group. (Exercise: Show that in a semigroup with the identity, if every element has a left inverse then every element is invertible and it is a group).

Conversely, suppose $T^{n_k}(x) \to c$ and $T^{n_k}(y) \to c$ for some $x, y, c \in X$ and an unbounded sequence $\{n_k\}$, then given a limit point S of $\{T^{n_k}\}$ there exists a sequence $\{m_k\}$ of $\{n_k\}$ such that $T^{m_k}(x) \to S(x) = c$ and $T^{m_k}(y) \to S(y) = c$. But as $E(T)$ is a group, S is invertible and hence $x = y$. That is T is distal. □

The following is due to Ellis [11].

Corollary 1.2 *Let X be a compact Hausdorff space and $T : X \to X$ be a bijective continuous map. If T is distal, then X is a disjoint union of minimal closed T-invariant sets, namely, closure of the orbits.*

Proof For $x \in X$, we have $\overline{O_x} = \overline{\{T^n(x)\}}_{n\in\mathbb{Z}} = E(T)x$. We show that this is a minimal closed T-invariant set as T is distal. It is T-invariant since $TE(T)x = E(T)x$, as $E(T)$ is a group. Moreover, for any $x' \in E(T)x$, $x' = Sx$ for some $S \in E(T)$. Then $E(T)x'$ is the smallest closed T-invariant subset of $E(T)x$ containing x', but $E(T)x' = E(T)Sx = E(T)x$ as $E(T)$ is a group. Now X is a disjoint union of $E(T)x$ as for any pair of elements $x, y \in X$, either $E(T)x = E(T)y$ or $E(T)x \cap E(T)y = \emptyset$. □

Definition 1.3 Let (X, d) be a metric space and $T : X \to X$ be bijective. T is said to be *equicontinuous* if the following holds: For any $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $d(T^n(x), T^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$.

Any equicontinous map is distal. Converse is not true even for a minimal flow.

Definition 1.4 A bijective continuous map $T : X \to X$ is said to be *minimal* if for any $x \in X$, $\{T^n(x)\}_{n\in\mathbb{Z}}$ is dense in X, i.e. there is no proper closed T-invariant set.

The following is well-known and can be found in Auslander's mathematical notes [2].

Theorem 1.5 *On a compact (metric) space X, any minimal flow T is equicontinuous if and only if $E(T)$ is a topological group.*

In particular, if T is minimal and equicontinuous, then $\{T^n\}_{n\in\mathbb{Z}}$ is dense in $E(T)$, which is a topological group, and hence $E(T)$ is abelian. Moreover, as T is minimal, $X = E(T)x$ for every $x \in X$. Fix any $x \in X$ and let $H = \{S \in E(T) \mid S(x) = x\}$, the stabiliser of x in $E(T)$. Then H is a closed subgroup of $E(T)$. Since $E(T)$ is abelian, and $X = E(T)x$, H stabilises every element of X and $E(T)/H$ is an abelian group, moreover, X is isomorphic to $E(T)/H$.

Corollary 1.6 *A compact (metric) space admits a minimal equicontinous flow if and only if it is an abelian topological group and T is a translation (shift) by an element.*

Proof Define a map $\phi : E(T) \to X$ as $\phi(S) = Sx$. As $E(T)$ is a topological group, ϕ is continuous and onto. Since H stabilises every element of X, we have that there is a continuous map $\bar{\phi} : E(T)/H \to X$ $\bar{\phi}(SH) = Sx$. Now $\bar{\phi}$ is a continuous bijection, and hence, a homeomorphism. Now T on X corresponds to a translation (shift) by T on $E(T)/H$. □

Have a look at Furstenberg's structure theorem for distal minimal transformation on compact metric spaces in [14]. It classifies all distal maps of a compact metric space. Any distal flow is a successive equicontinuous extensions starting from one point flow. It has also been generalised to non-metric spaces.

Now we are going to restrict ourselves to automorphisms of a topological group.

Let G be a locally compact Hausdorff topological group with the identity e and $T \in \mathrm{Aut}(G)$. I.e. T is a bijective bi-continuous homomorphism. Then T is distal if and only if for every $x \neq e$,

$$e \notin \overline{\{T^n(x) \mid n \in \mathbb{Z}\}}.$$

Throughout we will assume that our group is locally compact Hausdorff and second countable, (in particular, it is metrizable).

Example 4: Let $G = \mathbb{T}^2 = S^1 \times S^1$, 2-dimensional torus which is multiplicative group with the identity $(1, 1)$. Let $T(t_1, t_2) = (t_1t_2, t_2)$. It is easy to see that T is distal. Here, $T^n(t_1, t_2) = (t_1t_2^n, t_2)$. so if $T^{n_k}(t_1, t_2) = (t_1t_2^{n_k}, t_2) \to (e, e)$, where $e = (1, 0)$. Then $t_2 = e$ and hence $t_1 = e$. □

Example 5: Suppose G is a compact (non-abelian) group. Let $k \in G$ and take T to be the inner automorphism by k. I.e. $T(g) = kgk^{-1}$. Then T is distal, in fact $T^n(g) = k^ngk^{-n}$ for all $n \in \mathbb{Z}$. Then its limits points are of the from xgx^{-1} where x is a limit point of $\{k^n \mid n \in Z\}$. So $xgx^{-1} = e$ if and only if $g = e$. More generally, if K is a subgroup of G and if T is an inner automorphism of G by an element of k then T is distal. □

Example 6: Inner automorphisms of a nilpotent group (for e.g. G is the group of strictly upper-triangular matrices).

Let us define the *contraction group* of T, $C(T) = \{x \in G \mid T^n(x) \to e\}$. If T is distal then by definition, $C(T) = \{e\} = C(T^{-1})$. What about the converse?

This is easily true for $\mathbb{R}^d$. As $T \in GL(d, \mathbb{R})$ and $C(T)$ is a T-invariant subspace and the eigenvalues of T restricted to $C(T)$ are of absolute value less than 1 unless $C(T)$ is trivial. The converse is true for all locally compact group but we will not go into the detail here.

The first question we discuss is wether distality carries over if you go to the quotients modulo a compact invariant subgroup. The following is a joint work with C.R.E. Raja (cf. [22]).

Theorem 1.7 *Suppose K is a compact (normal) T-invariant subgroup of G for some $T \in \mathrm{Aut}(G)$. Let $\bar{T} \in \mathrm{Aut}(G/K)$ be the corresponding map defined as $\bar{T}(xK) = T(x)K$, for $x \in G$. Then the following holds: T is distal on G if and only if $\bar{T}$ is distal and $T|_K$, the restriction of T to K, is also distal.*

Proof Suppose $\bar{T}$ is distal and $T|_K$ is distal. For some $x \in G$, suppose $T^{n_k}(x) \to e$ for some sequence $\{n_k\}$. Then $T^{n_k}(x)K \to eK = K$ in G/K. Hence $\bar{T}^{n_k}(xK) \to K$ in G/K. Since $\bar{T}$ is distal, it implies that $xK = K$ i.e. $x \in K$. Since $T|_K$ is distal, we have $x = e$. Hence T is distal.

Conversely, assume that T is distal on G. Then its restriction to K is obviously distal. We show that $\bar{T}$ is distal. Suppose $(\bar{T})^{n_k}(xK) = T^{n_k}(x)K \to K$ for some $x \in G$. We have to show that $xK = K$, i.e. $x \in K$. This implies that $T^{n_k}(x) = u_k h_k$ for some $u_k \to e$ and $\{h_k\} \subset K$. Since K is compact, passing to a subsequence if necessary, we may assume that $h_k \to h \in K$ and hence $T^{n_k}(x) \to h$. Since $T|_K$ is distal, $E(T|_K)$ is a group in K^K (by Theorem 1.1). Let $\gamma \in E(T|_K)$ be the limit point of $\{(T|_K)^{n_k}\}$. Let $b = \gamma^{-1}(h) \in K$. Passing to a subsequence if necessary, we may assume that $T^{n_k}(b) \to \gamma(b) = h$. Since T is distal on G, $x = b \in K$, i.e. $xK = K$ and $\bar{T}$ is distal. □

The above theorem allows us to go to quotient by compact normal invariant subgroups. Any connected group has a maximal compact normal (characteristic) subgroup such that G/K is a (connected) Lie group.

Let G^0 be the connected component of the identity e, i.e. it is the maximal connected subset of G containing e. Then G^0 is a closed normal subgroup of G.

A topological space is said to be *totally disconnected* if its connected components are singleton sets. For a topological group G, it is totally connected iff $G^0 = \{e\}$. In particular for a group G, the quotient group G/G^0 is totally disconnected.

Examples of totally disconnected groups: Any group with discrete topology, finite groups, $\mathbb{Z}$, $GL(n, \mathbb{Z})$, for a prime p, the field of p-adic numbers $\mathbb{Q}_p$ with addition as operation, closed subgroups of $GL_n(\mathbb{Q}_p)$. Any (closed) subgroup of a totally disconnected group is totally disconnected.

If G is a totally disconnected locally compact group, then it has a neighbourhood base of the identity e consisting of compact open subgroups.

The following by Jaworski and Raja characterises distal actions on totally connected groups (cf. [16]).

Theorem 1.8 *Let G be a totally disconnected locally compact group and let $T \in \mathrm{Aut}(G)$. Then the following are equivalent:*

1. *T is distal*

2. *$C(T) = C(T^{-1}) = \{e\}$.*

3. G has a neibhgourhood bases of T-invariant compact open subgroups.

Our next question is if distality is preserved if we go modulo the connected component of the identity e. The following is a joint work with C.R.E. Raja (cf. [22]).

Theorem 1.9 *For $T \in \mathrm{Aut}(G)$, let $\bar{T} : G/G^0 \to G/G^0$ is the automorphism defined from T, i.e. $\bar{T}(xG^0) = T(x)G^0$ for all $x \in G$. Then T is distal if and only if $T|_{G^0}$ is distal and the $\bar{T}$ on G/G^0 is distal.*

Proof: The 'if' part follows as in the proof of the previous theorem. Now we prove the 'only if' part. Suppose T is distal on G. Then its restriction to G^0 is distal. We want to prove the distality of $\bar{T}$. We will show that $C(\bar{T})$ is trivial. Suppose $\bar{T}^n(xG^0) \to G^0$ in G/G^0. From Theorem 1.7, we can assume going modulo the maximal compact normal subgroup of G^0, that G^0 is a group without any non-trivial compact normal subgroup. From the structure theory of locally compact groups, we can find a compact totally disconnected subgroup K of G such that K centralises G^0 and $G^0 \times K = H$ is open in G. Since $T^n(x)G^0 \to G^0$, $T^n(x) \in H$ for large n, and hence replacing x by $T^n(x)$, we may assume that $x \in K \times G^0$ and $T^n(x) \in K \times G^0$. Let $x = kg = gk$, where $k \in K$ and $g \in G^0$. Then $T^n(k)G^0 \to G^0$. Since K is totally disconnected and it centralises G^0 and G^0 is a group without any non-trival compact normal subgroup we have that $T^n(k) \in K$ and moreover, as $T^n(k)G^0 \to G^0$, we can show that $T^n(k) \to e$ and hence $k = e$. Then $x = g \in G^0$. I.e. $C(\bar{T})$ is trivial. Similarly, $C(\bar{T}^{-1})$ is trivial and hence $\bar{T}$ is distal from Theorem 1.8.

2 Distal Groups

In this section, we define distal and pointwise distal groups and discuss their properties: Let G be a locally compact (second countable) group.

G is said to be *pointwise distal* if the inner automorphism by every element in G is distal. Equivalently, for every $g \in G$, and for every $x \neq e$.

$$e \notin \overline{\{g^n x g^{-n} \mid n \in \mathbb{Z}\}}.$$

G is said to be *distal* if the conjugacy action of G on G is distal. Equivalently, for every $x \neq e$,

$$e \notin \overline{\{g x g^{-1} \mid g \in G\}}.$$

Example 7: Abelian groups, here the conjugacy action of each element is the identity map. More generally, nilpotent groups.

Example 8: Compact groups: $\{gxg^{-1} \mid g \in G\}$ is closed (and compact) and it doesn't contain the identity e unless $x = e$.

Example 9: Discrete groups: If $g_n x g_n^{-1} \to e$, then $g_n x g_n^{-1} = e$ for large n and hence $x = e$.

Example 10: If G is a compact extension of a nilpotent (normal) subgroup. I.e. if G has a closed (normal) nilpotent subgroup N such that G/N is compact. More generally,

Lemma 2.1 *If H is a closed subgroup of G such that H is distal and G/H is compact then G is distal.*

Not all groups are (pointwise) distal. For e.g. non-compact semisimple groups for e.g. $SL_n(\mathbb{R})$. Not all solvable groups are pointwise distal.

Example 11: Let $G = \mathbb{R}_+^* \ltimes \mathbb{R}$, whose group operation is defined as $(a,b)(a',b') = (aa', ab' + b)$, and the identity of G is $(1,0)$. If $g = (1/2, 0)$ and $x = (1,1)$, then $g^n x g^{-n} = (1, 1/2^n) \to (1,0)$ as $n \to \infty$. G is not (pointwise) distal.

The following two corollaries follow from Theorems 1.7 and 1.9 respectively (cf. [22]).

Corollary 2.2 *A group G is (pointwise) distal if and only if for any compact normal subgroup K, G/K is (pointwise distal) and the conjugacty action of G on K is (pointwise) distal.*

Corollary 2.3 *A group G is (pointwise) distal if and only if G/G^0 is (pointwise) distal and for every $g \in G$, the conjugacy action of g restricted to G^0 is distal.*

Subgroups of distal groups are distal. What about quotients: It is true from above corollaries that quotients of distal groups are distal if they are modulo compact group or connected components. (Or if the quotient is compact or discrete or nilpotent, it is anyway distal).

Quotients modulo closed normal subgroup of pointwsie distal groups are pointwise distal. We do not know if the corresponding statement for the quotients of distal groups, except in some obvious cases.

Distal groups are obviously pointwise distal but the converse is not true as is shown by the following example which is well-known:

Example 12: Let $G = S \ltimes K$, a semidirect product of groups, where $S = \cup_{n=1}^{\infty} S_n$, S_n is the group of all permutations of order n, and $K = \Pi_{i=1}^{\infty} K_i$, $K_i = C$, a nontrivial compact group for all i, for e.g. $C = S^1$. Here, S is endowed with discrete topology and K, being a product of compact groups, is compact. The action of S on K is given as follows: If α is a permutation of oder n, then

$$\alpha(k_1, \ldots, k_n, k_{n+1} \ldots)\alpha^{-1} = (k_{\alpha(1)}, \ldots, k_{\alpha(n)}, k_{n+1}, \ldots).$$

Since G/K is discrete, it is enough to consider the congugacy action of G on K. Here, K is distal, any element α in S has finite order as an automorphism of K. Hence it acts distally on K. But if you take an element $k = (a, e, ...)$, $a \neq e$ in C, and take $\alpha_n = (1, n)$, then $\alpha_n k \alpha_n^{-1} = c_n$ whose all entries are e except for the nth entry which is a. So we have a non-trivial element k whose orbit closure contains the identity e. □

For (almost) connected groups G, it has been shown by Rosenblatt that G is distal if and only if G is pointwise distal.

3 Concentration functions of a probability measure and point-wise distal groups

In this section, we are going to relate pointwise distal groups and behaviour of convolution powers of a probability measures; namely, the behaviour of concentration functions of a probability measure.

Let G be locally compact group. Let $P(G)$ denote the set of all (regular Borel) probability measures on G. Then $P(G)$ has a semigroup structure give by the convolution operation '$*$'. If $\mu, \lambda \in P(G)$ are two probability measures then $\mu * \lambda = o(\mu \times \lambda)$ where $o : G \times G \to G$ is the group operation $o(x, y) = xy$. i.e.

$$\mu * \lambda(B) = \mu \times \lambda(\{(x, y) \mid xy \in B\}) = \int_G \mu(By^{-1}) \, \mathrm{d}\lambda(y) = \int_G \lambda(x^{-1}B) \, \mathrm{d}\mu(x).$$

On $\mathbb{R}$ for e.g. we will have

$$\mu * \lambda(B) = \mu \times \lambda(\{(x, y) \mid x + y \in B\}).$$

$P(G)$ is semigroup with the identity δ_e, the Dirac measure at the identity e of the group, where δ_e is defined as $\delta_e(B) = 1$ if $e \in B$ otherwise it is zero for any Borel set B. $P(G)$ is a topological semigroup with weak topology, it is the smallest topology on $P(G)$, which makes $\mu_\alpha \to \mu$, if and only if $\mu_\alpha(f) \to \mu(f)$ for all bounded continuous real valued functions f on G.

One can embed G in $P(G)$ as $g \mapsto \delta_g$, the Dirac measure at g, and $\delta_g * \delta_h = \delta_{gh}$, for $g, h \in G$.

For any $\mu \in P(G)$, we define $\mu^n = \mu * \cdots * \mu$ (n-times).

Now we define *concentration functions* of a measure.

Definition 3.1 For $\mu \in P(G)$, given a compact set K in G and $n \in \mathbb{N}$, the *nth concentration function of* μ is

$$f_n(\mu, K) = \sup_{x \in G} \mu^n(Kx^{-1}).$$

Naturally, since $\mu^n \in P(G)$, $f_n(\mu, K)$ is bounded below by 0 and bounded above by 1.

Moreover, $\{f_n(\mu, K)\}$ is a non-increasing (monotone) sequence for a given compact set K. For any $x \in G$,

$$\mu^{n+1}(Kx^{-1}) = \mu^n * \mu(Kx^{-1}) = \int_G \mu(Kx^{-1}y^{-1})\, \mathrm{d}\mu(y) \leq \int_G f_n(\mu, K)\, \mathrm{d}\mu(y) = f_n(\mu, K).$$

Since the above is true for all $x \in G$, for all $n \in \mathbb{N}$,

$$f_{n+1}(\mu, K) = \sup_{x \in G} \mu^{n+1}(Kx^{-1}) \leq f_n(\mu, K).$$

We say that the concentration functions of μ converge to zero if for every compact set K in G, $f_n(\mu, K) \to 0$. Otherwise, we say that they do not converge to zero. i.e. there exists a compact set K for which it does not converge to zero.

Let us see some examples.

Example 13: Let G be any group (for e.g. $G = \mathbb{R}$ or $\mathbb{R}^d$) and $\mu = \delta_g$ for some $g \in G$. Then $\mu^n = \delta_{ng}$. Then for any compact set K, for $x = g^{-n}k$ for some $k \in K$, we have $g^n \in Kx^{-1}$ and hence

$$\mu^n(Kx^{-1}) = \delta_{g^n}(Kx^{-1}) = 1.$$

Hence $f_n(\mu, K) = 1$ for all n. The concentration functions of μ do not converge to zero.

Example 14: Let G be a group, H be a compact subgroup and let $g \in G$ be such that $gHg^{-1} = H$. Let $\mu = \delta_g * \omega_H$ $(= \omega_H * \delta_g)$. Then $\mu^n = \delta_{g^n} * \omega_H$ and $f_n(\mu, H) = 1$ for all n.

Example 15: Let G be a compact group. Then $P(G)$ is compact and the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ is relatively compact, hence for every $\epsilon > 0$, there exists a compact set K_ϵ such that $\mu^n(K_\epsilon) > 1 - \epsilon$. Then $f_n(\mu, K_\epsilon) > 1 - \epsilon$.

Example 16: Let G be a discrete group. Then for any measure μ, $\{f_n(\mu, K)\}$ converge to zero unless $\mu = \delta_g * \lambda$ where λ is supported on a compact (finite) subgroup normalised by g.

Example 17: Let $G = \mathbb{R}$. Suppose μ is such that the support of μ has at least two elements in it, for e.g. $\mu = (\delta_0 + \delta_1)/2$. Then the concentration functions of μ do not converge to zero.

If the concentration functions of μ do not converge to zero, then there exists an element $x \in G$, such that $\{\mu^n * \delta_x^{-n}\}$ is relatively compact in $P(G)$.

In Example 2, $\mu^n * \delta_{g^{-n}} = \omega_H$.

The following result gives a structure of the group generated by the support of μ in case the concentration functions of μ do not converge to zero. It was obtained jointly with S.G. Dani for Lie groups (cf. citeDS, and it is due to Jaworski, Rosenblatt and Willis for all locally compact groups (cf. [17]).

Theorem 3.2 *Let G be a locally compact group and $\mu \in P(G)$ be such that the concentration functions of μ do not converge to zero. Then there exists subgroups $H \subset M$, H compact and an element $x \in$ suppμ which normalises both H and M such that* supp$\mu \subset xM = Mx$ *and the conjugacy action of x on M/H contracts M/H. i.e. for all $g \in M$, $x^n g x^{-n} H \to H$ in G/M.*

This, in particular, gives the structure of $G(\mu)$, the closed subgroup generated by the support of μ. $G(\mu) = \mathbb{Z} \ltimes M$, $G(\mu)/H = \mathbb{Z} \ltimes M/H$ and for $x = 1$, the generating element of Z, the conjugacy action of x on M/H contracts whole of it.

We say that a group G has shifted convolution property if given any $\mu \in P(G)$, either the concentration functions of μ converges to zero or there exist a compact subgroup H and an element $x \in G$ which normalises H such that $\mu^n * \delta_{x^{-n}} \to \omega_H$.

Note that in the second case, we have that supp$\mu \subset xH = Hx$ and hence for every $g \in$ suppμ, $\mu^n * \delta_{g^{-n}} \to \omega_H$.

The following is joint work with C.R.E. Raja (cf. [22]).

Theorem 3.3 *A locally compact group is point-wise distal if and only if it has shifted convolution property.*

Idea of Proof for the 'if' part: Suppose G is pointwise distal. Let $\mu \in P(G)$ be such that the concentration functions of μ do not converge to zero. Then $G(\mu)$, the closed subgroup generated by the support of μ is also point-wise distal. From Theorem 3.2, it has a compact normal subgroup H and $G(\mu)/H = Z \ltimes M/H$ and the conjugacy action of $x = 1$ in $\mathbb{Z}$ contracts M/H. If $H \neq M$, $G(\mu)/H$ is not pointwise distal, As H is compact, by Corollary 2.2, $G(\mu)$ itself is not pointwise distal, a contradiction. Hence, $H = M$ is compact. Hence $\mu = \delta_x * \lambda$, where λ is supported on the compact group H. Now since $G(\mu) = \mathbb{Z} \ltimes H$ is pointwise distal, one can show that $\mu^n x^{-n} \to \omega_H$.

4 Distality and Minimal Orbit Closures

For an automorphism T of G, and $x \in G$, we call $O_x = \{T^n(x) \mid n \in \mathbb{Z}\}$, the orbit of x. A map T on a group G is said to have the property [MOC] if for every $x \in X$, the orbit closure $\overline{\{T^n(x) : n \in \mathbb{Z}\}}$ is a minimal closed T-invariant set. I.e. there is no proper closed T-invariant subset of $C_x = \overline{O}_x = \overline{\{T^n(x)\}}_{n \in \mathbb{Z}}$. This is equivalent to the following: For any $x, y \in G$, if $y \in C_x$, then $C_y = C_x$ (i.e. $x \in C_y$); In other words, if $T^{n_k}(x) \to y$ for some sequence $\{n_k\}$, then $T^{m_k}(y) \to x$ for some sequence $\{m_k\}$).

[MOC] stands for '*minimal orbit closures*'.

If T has [MOC], then G splits into disjoint minimal closed T-invariant sets.

If T has [MOC], then T is distal. (For if $T^{n_k}(x) \to e$ for some $x \neq e$. Then C_x is not minimal as $\{e\}$ is a proper closed T-invariant subset of C_x.)

The above holds also for an action of any subgroup of Aut(G).

What about the converse?

For a compact group, distality and [MOC] are equivalent, (see Corollary 1.2).

If G is a discrete group, then any set is closed and hence the orbit of x, $O_x = C_x$. Therefore, if $y \in C_x$, then $y = T^m(x)$ for some m and hence $x = T^{-m}(y) \in O_y = C_y$, hence, C_x is a minimal closed T-invariant set.

Let $G = \mathbb{R}^d$ and let $T \in \mathrm{Aut}(G)$ be distal. Suppose $T^{n_k}(x) \to y$ for some $x, y \in \mathbb{R}^d$. Take $W = \{w \in \mathbb{R}^d \mid T^{n_k}(w) \text{ converges}\}$. Then W is a closed T-invariant subspace and $x \in W$. Since W is T-invariant and closed, it contains $T^n(x)$ for all n and also y. So W is non-trivial. Moreover, $T^{n_k}|_W$, the restriction of T^{n_k} to W converges point-wise, we have that $T^{n_k}|_W$ converges to (say) ψ, which is a homomorphism. Since T is distal, ψ injective and hence $\psi \in \mathrm{Aut}(W) = GL(W)$. Then $y = \psi(x)$ and $T^{-n_k}|_W \to \psi^{-1}$. Hence $T^{-n_k}(y) \to \psi^{-1}(y) = x$.

On a connected Lie group, distality and [MOC] are equivalent (cf. Abels). in fact, Abels proved it for an action of any automorphism group.

If G is totally disconnected then T is distal implies T has [MOC] (this follows from Theorem 2.3, see [25]). This is because, G has a neighbourhood bases of the identity e consisting of T-invariant compact open subgroups say K_α. Then G/K_α is a discrete space and the corresponding maps $T_\alpha : G/K_\alpha \to G/K_\alpha$ is distal and also has [MOC] as the orbits are closed. This implies that $C_x K_\alpha = O_x K_\alpha$ for all $x \in G$ and every K_α. if $x \in G$, and $y \in C_x$, then $C_x K_\alpha = C_y K_\alpha$ for all K_α. Since $\cap_\alpha K_\alpha = \{e\}$. We get that $C_x = C_y$.

The following are proven in [25].

Theorem 4.1 *Let G be locally compact group and $T \in \mathrm{Aut}(G)$. Let K be a compact (normal) T-invariant subgroup of G. Then T has* [MOC] if and only if $T|_K$ has [MOC] and $\bar{T}$ on G/K has [MOC].

Theorem 4.2 *Let G be a locally compact group and $T \in \mathrm{Aut}(G)$. Then T is distal if and only if T has* [MOC].

The above are also true for the action of a locally compact compactly generated abelian group of automorphisms.

References

[1] H. Abels, Distal automorphism groups of Lie groups, J. Reine Angew. Math. 329 (1981), 82–87.

[2] J. Auslander, Minimal flows and their extensions, Notas de Matematica [Mathematical Notes], 122, North-Holland Mathematics Studies, vol. 153, North-Holland Publishing Co., Amsterdam, 1988.

[3] W. Bartoszek, On concentrated probabilities. Ann. Polon. Math. 61 (1995), no. 1, 25–38.

[4] J. P. Conze and Y. Guivarc'h, Remarques sur la distalit dans les espaces vectoriels, C. R. Acad. Sci. Paris Sr. A 278 (1974), 1083–1086.

[5] I. Csiszár, On infinite products of random elements and infinite convolutions of probability distributions on locally compact groups, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 5 (1966), 279–295.

[6] S. G. Dani, On automorphism groups of connected Lie groups. Manuscripta Math. 74 (1992), no. 4, 445–452.

[7] S. G. Dani, and C. R. E. Raja, Asymptotics of measures under group automorphisms and an application to factor sets. Lie groups and ergodic theory (Mumbai, 1996), 59–73, Tata Inst. Fund. Res. Stud. Math., 14, Tata Inst. Fund. Res., Bombay, 1998.

[8] S. G. Dani and C. R. E. Raja, A note on Tortrat groups. J. Theoret. Probab. 11 (1998), no. 2, 571–576.

[9] S. G. Dani and R. Shah, Collapsible probability measures and concentration functions on Lie groups. Math. Proc. Cambridge Philos. Soc. 122 (1997), no. 1, 105–113.

[10] Y. Derriennic and M. Lin, Convergence of iterates of averages of certain operator representations and of convolution powers. J. Funct. Anal. 85 (1989), no. 1, 86–102.

[11] P. Eisele, On shifted convolution powers of a probability measure, Math. Z. 211 (1992), no. 4, 557–574.

[12] P. Eisele, Tortrat groups—groups with a nice behavior for sequences of shifted convolution powers. J. Theoret. Probab. 6 (1993), no. 4, 671–691.

[13] R. Ellis, Distal transformation groups, Pacific J. Math. 8 (1958), 401–405.

[14] H. Furstenberg, The structure of distal flows. Amer. J. Math. 85 (1963), 477–515.

[15] H. Heyer, Probability measures on locally compact groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94. Springer-Verlag, Berlin-New York, 1977.

[16] W. Jaworksi and C. R. E. Raja, The Choquet-Deny theorem and distal properties of totally disconnected locally compact groups of polynomial growth, New York J. Math. 13 (2007), 159-174.

[17] W. Jaworski, J. Rosenblatt and G. Willis, Concentration functions in locally compact groups. Math. Ann. 305 (1996), 673–691.

[18] B. Kitchens and K. Schmidt, Automorphisms of compact groups. Ergodic Theory Dynam. Systems 9 (1989), no. 4, 691–735.

[19] C. R. E. Raja, On classes of p-adic Lie groups. New York J. Math. 5 (1999), 101–105.

[20] C. R. E. Raja, On growth, recurrence and the Choquet-Deny theorem for p-adic Lie groups. Math. Zeit. 251 (2005), no. 4, 827–847.

[21] C. R. E. Raja, Distal actions and ergodic actions on compact groups, New York J. Math. 15 (2009), 301–318.

[22] C. R. E. Raja and Riddhi Shah, Distal actions and shifted convolution property, Israel J. Math. 177 (2010), 391–412.

[23] J. Rosenblatt, A distal property of groups and the growth of connected locally compact groups, Mathematika 26 (1979), no. 1, 94–98.

[24] K. Schmidt, Dynamical systems of algebraic origin. Progress in Mathematics, 128. Birkhuser Verlag, Basel, 1995.

[25] Riddhi Shah, Orbits of distal actions on locally compact groups, J. Lie Theory 22 (2012), 587–599.

[26] E. Siebert, Contractive automorphisms on locally compact groups, Math. Z. 191 (1986), no. 1, 73–90.

[27] E. Siebert, Semistable convolution semigroups and the topology of contraction groups. Probability measures on groups, IX (Oberwolfach, 1988), 325–343, Lecture Notes in Math., 1379, Springer, Berlin, 1989.